

Option Prices and Option-Implied Probabilities in Incomplete Markets

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Abstract

This paper develops a methodology to quantify the uncertainty about option prices that persists in (statically) incomplete option markets due to finite strike and maturity grids as well as bid-ask price quoting. Uncertainty measures are provided for European option prices and their slopes (i.e., tail probabilities), each derived from model-free bounds that are consistent with a finite sample of observed bid and ask option prices and elementary no-arbitrage constraints in a multi-period setting. We obtain novel explicit expressions for these bounds, which are simple and efficient to compute. Moreover, we suggest a decomposition of the bounds and uncertainty measures into contributions of strike, maturity, and price quote incompleteness. In an empirical analysis of S&P 500 index options, we quantify the empirical uncertainty about option prices. We document sizable uncertainty levels for option prices and tail probabilities. Our findings challenge the common belief that such "option-implied" (tail) information can be measured at sufficiently high precision, which has implications for many applications that use this information.

JEL classification: C14, C61, G12, G13

Keywords: European options; incomplete markets; option-implied moments; model-free bounds; uncertainty quantification

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1 Introduction

Throughout this paper, we are concerned with the distributional information embedded in the prices of plain-vanilla European options written on some underlying asset price. When only finitely many option prices are observed, potentially only in the form of bid and ask prices, — as is the case in empirical data sets — it is typically possible to find infinitely many risk-neutral probability measures that are consistent with these observed prices.¹ Nevertheless, it is a common conception in the academic research community that option prices and ”option-implied” probability distributions can be measured with high precision from the observed option data. This paper challenges this common conception by quantifying and analyzing the actual uncertainty about option prices and certain ”option-implied” probabilities that arises in a (statically) incomplete option market with finitely many traded strikes and maturities as well as bid-ask price quoting.

As its first methodological contribution, our paper provides model-free bounds for arbitrage-free option prices within such an incomplete market setting. To formally derive optimal bounds, we set up and solve certain generalized moment problems.² In essence, we determine lower and upper option price bounds over all probability measures that are consistent with a finite number of observed bid and ask option prices across multiple maturities, imposing only minimal structure through elementary no-arbitrage conditions. The obtained bounds are simple and efficient to compute, effectively through linear interpolation and extrapolation rules involving the imposed bid-ask constraints. The width of these bounds, i.e., the difference between the respective upper and lower bounds, yields a measure of the uncertainty about the corresponding ”true” option price. Moreover, we devise an approach to decompose the bounds and total uncertainty into contributions of incompleteness in the maturity dimension (due to discrete maturity grids), in the strike dimension (due to discrete or bounded strike grids), and in the quoted price dimension (due to bid-ask spreads).

Subsequently, as its second methodological contribution, our paper also derives model-free bounds for left-tail and right-tail probabilities of the underlying asset price within an analogous setting. Essentially, such tail probabilities correspond to the (directional) slope of option prices in the strike dimension. We determine explicit lower and upper bounds for these tail probabilities, which may be expressed entirely in terms of finite-difference quotients involving the derived option price bounds, making them again simple and efficient to compute. Analogous to the case

¹This situation differs fundamentally from the ideal case of a perfectly complete market, where a continuum of options is traded for any maturity and (frictionless) option prices allow to uniquely recover an associated ”option-implied” risk-neutral probability measure. The latter case has been well-understood for a long time, at least since the seminal contributions of Banz and Miller (1978) and Breeden and Litzenberger (1978).

²In their generic form, generalized moment problems are infinite-dimensional linear programs used to determine lower and upper bounds for some specified target moment over the set of positive Borel measures, constrained by a set of moment conditions. Generalized moment problems are extensions of the classical moment problems, which are concerned with the existence and uniqueness of measures given a sequence of values of (typically polynomial) moments (e.g., Schmüdgen (2017)). An introduction to generalized moment problems can be found in, e.g., Lasserre (2009) and Shapiro (2001). A comprehensive account of the general theory of infinite-dimensional linear programming is provided by, e.g., Anderson and Nash (1987).

of option prices, the width of these bounds quantifies the total uncertainty about "true" tail probabilities, which we likewise decompose into contributions of incompleteness in the maturity, strike, and quoted price dimensions.

In the empirical part of this paper, we apply our methodology to a sample of S&P 500 index options. Specifically, we quantify option price and tail probability bounds at a constant (monthly) maturity, which corresponds to the usual setting of most of the existing empirical research. In this context, "option-implied" information is in particular exposed to maturity incompleteness, as one is interested in measuring quantities at a constant maturity, while the observed option maturities are time-varying. For option price bounds, we accordingly find a dominating role of the uncertainty due to maturity incompleteness, which explains on average about 90% of the observed uncertainty. A large contribution of maturity incompleteness is plausible, since no-arbitrage constraints in the maturity dimension are rather weak. Apart from this maturity component, we observe relatively small uncertainty at medium moneyness levels, while the uncertainty at more extreme moneyness levels can still be large especially during the earlier part of the sample, as the evaluation takes place outside of the observed strike range. For tail probability bounds, we similarly obtain that maturity incompleteness can explain a sizable, but much smaller, fraction of the total uncertainty. Substantial uncertainty remains at all moneyness levels even when removing this component. This is as tail probabilities essentially require to determine the slope of option prices in the strike dimension, using only observed level information contained in bid and ask prices. In fact, when considering moneyness levels scaled by a time-varying volatility measure, the uncertainty about tail probability measures, even at medium moneyness levels, tends to dominate any structural time series behavior (i.e., the bounds span an almost constant corridor).

Our analysis of the uncertainty conveyed by option price and tail probability bounds sheds new light on empirical research that aims at establishing an association between "option-implied" (tail) information and other economic measures. E.g., Gormsen and Jensen (2025) observe a counter-cyclical time series behavior of certain option-based tail probability measures. In contrast, the width of tail probability bounds in our analysis suggests that — at least for a vast part of the sample — observed option prices are consistent with any (almost) arbitrary time series behavior, including cyclical and counter-cyclical. As our analysis so far only covers the most liquid S&P 500 index, the issues already identified here can be expected to be even much more severe in less developed option markets.

On the methodological side, some existing literature related to our work also aims at providing bounds for option prices in an incomplete market setting.³ Although their work is not immediately concerned with option price bounds as in this paper, our work builds on techniques and results

³While our paper is effectively situated in a static setting, related methodologies are also encountered when formalizing no-arbitrage conditions in dynamic or semi-static settings (e.g., Acciaio et al. (2016), Dolinsky and Neufeld (2017), and Hou and Oblój (2018)). As in the static setting (e.g., Hobson and Klimmek (2015), Hobson and Neuberger (2012), and Neufeld et al. (2023)), an active related research area in dynamic or semi-static settings concerns model-free bounds for various exotic derivatives (e.g., Beiglböck et al. (2013), Eckstein et al. (2021), and Kahalé (2017)). Our approach in this paper shares similar conditions to assure intertemporal consistency with parts of this literature, although under a lighter technical structure that allows for various explicit solutions.

in Davis and Hobson (2007), who essentially establish no-arbitrage conditions for option prices in a multi-period setting. Similar results are provided in, e.g., Carr and Madan (2005) and Cousot (2007), with recent generalizations in Gerhold and Gülüm (2020). Most closely related to the agenda of our paper is the contribution of Bertsimas and Popescu (2002), who — among a large number of different cases — provide explicit model-free bounds for option prices in a single-period context using a generalized moment problem setup. Notably, for the case of constraints imposed by a finite number of options with non-trivial bid-ask spreads, their bounds substantially differ from those derived in this paper. In addition to that, we significantly generalize their formulation to a tractable multi-period setup.

Related to our work is also the literature that derives option price bounds using exogenous information about the distribution of the underlying asset price. Employing a single-period generalized moment problem setup, Henrion et al. (2021) develop a computational methodology to provide model-free bounds for option prices subject to option price constraints together with additional exogenous constraints on certain higher-order moments of the underlying price. In a multi-period setting, Primbs (2006, 2010) instead impose constraints on piecewise-polynomial moments of the underlying price over different time horizons. These approaches are connected to a line of research that exclusively considers higher-order moment constraints, historically dating back to at least Lo (1987). Different variants of such problems are also investigated in Bertsimas and Popescu (2002) as well as, among others, Boyle and Lin (1997), Gotoh and Konno (2002), Grundy (1991), Ritchken and Kuo (1988), and Zuluaga et al. (2009). Complementary to these techniques, stochastic dominance approaches incorporate assumptions on risk preferences that shape the stochastic discount factor (e.g., Chazal and Jouini (2008), Constantinides and Perrakis (2002), Levy (1985), Perrakis and Ryan (1984), Ritchken (1985), Ritchken and Kuo (1989), and Ryan (2003)). Relatedly, Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000) also impose certain economic restrictions on stochastic discount factors. More recently, Almeida and Freire (2022) determine option price bounds over a parameterized family of minimum divergence measures. What these latter approaches have in common is that they require some exogenous prior information about the distribution of the underlying asset price or the stochastic discount factor, which needs to be estimated or otherwise arbitrarily specified, while this paper only relies on actually observed option price information.

In a wider sense, our work is also related to a vast existing literature that deals with general "option-implied" moments, which are typically obtained from certain static portfolios involving a continuum of options (e.g., Britten-Jones and Neuberger (2000) and Carr and Madan (2001b)). This literature rarely, if at all, explicitly accounts for an incomplete option market, but instead employs interpolation and extrapolation assumptions to artificially construct completeness. By considering a realistic incomplete market setting, the approach followed in this paper constitutes a substantial methodological generalization relative to this literature, which eventually exposes several serious issues that are comprehensively investigated in our companion work in Bondarenko et al. (2024).

The remainder of this paper is organized as follows. Section 2 provides a motivating example.

Section 3 derives explicit expressions for the model-free bounds for option prices and tail probabilities and constructs the uncertainty measures. Sections 4 and 5 contain our empirical application to S&P 500 option data. Section 6 briefly discusses further implications. Finally, section 7 concludes the paper. Additional supporting material is contained in the appendix.

2 A motivating example

Consider an option market, in which for convenience options are written on the forward return R_t between times 0 and t . Given arbitrage-free call option prices $C(k, t)$ for any strike $k \in (0, \infty)$ and a particular maturity t , a generalization of the well-known Breeden and Litzenberger (1978) formula yields a unique characterization of the option-implied risk-neutral probability measure $\mathbb{Q}_t$ such that⁴

$$\mathbb{Q}_t(a \leq R_t \leq b) = \mathbb{E}^{\mathbb{Q}_t}[\mathbb{1}_{[a,b]}(R_t)] = \partial_{k+}C(b, t) - \partial_{k-}C(a, t) \quad (2.1)$$

for any $a \leq b$. Here, ∂_{k+} and ∂_{k-} denote the right-hand and left-hand directional derivative operators, respectively, with respect to the strike price. It should be noted that $\partial_{k+}C(b, t)$ and $\partial_{k-}C(a, t)$ are determined from the local variation of $C(k, t)$ over strike intervals $[b, b + \varepsilon)$ and $(a - \varepsilon, a]$ for arbitrarily small $\varepsilon > 0$.

We say that an option market is (statically) complete at time horizon t if observed option prices allow to uniquely recover the respective option-implied probability measure via equation (2.1), accounting for elementary (static) no-arbitrage restrictions. In general, this requires knowledge of the local behavior of arbitrage-free option prices $C(k, t)$ around *any* positive strike k at the given maturity t . In empirical option markets, we however encounter three notions of option market incompleteness, each of which individually impede this unique recovery:

- *Strike incompleteness*: Not every positive strike is traded. Realistic strike sets rather contain only finitely many elements, bounded away from both boundaries of the return domain.
- *Quote incompleteness*: Quoted bid and ask prices are observed instead of "true" option prices. The exact location of true prices within the bid-ask range cannot be inferred from the observed data.
- *Maturity incompleteness*: Not every positive maturity is traded. Realistic maturity sets are finite, typically with relatively few elements. Oftentimes arises when considering a constant time horizon, while observed maturity sets are time-varying.

In incomplete option markets, there thus exists a multitude of admissible risk-neutral probability measures that are consistent with observed option prices. Within this setting, the theoretical objective of this paper is to determine explicit model-free bounds for option prices and their directional derivatives (i.e., left-tail and right-tail probabilities) across any such admissible probability measures.

⁴Essentially, this expression follows from the general static replication formula in Bick (1982).

To illustrate the effects of option market incompleteness in a simplified setting, consider a Black-Scholes economy with an attached option market. In this market, out-of-the-money European options on R_t are traded for a single maturity and a discrete strike set, with prices observed in the form of (forward) bid and ask prices. For a concrete parameterization of this economy, Figure 2.1 plots the model-free bounds for option prices and tail probabilities obtained from our main theoretical results. The option price bounds in panel (a) are relatively narrow at the observed strike locations, while becoming considerably wider inbetween strikes and especially towards the tails. In contrast, the tail probability bounds in panel (b) are substantially wider throughout the entire strike range, even at the observed strike locations.

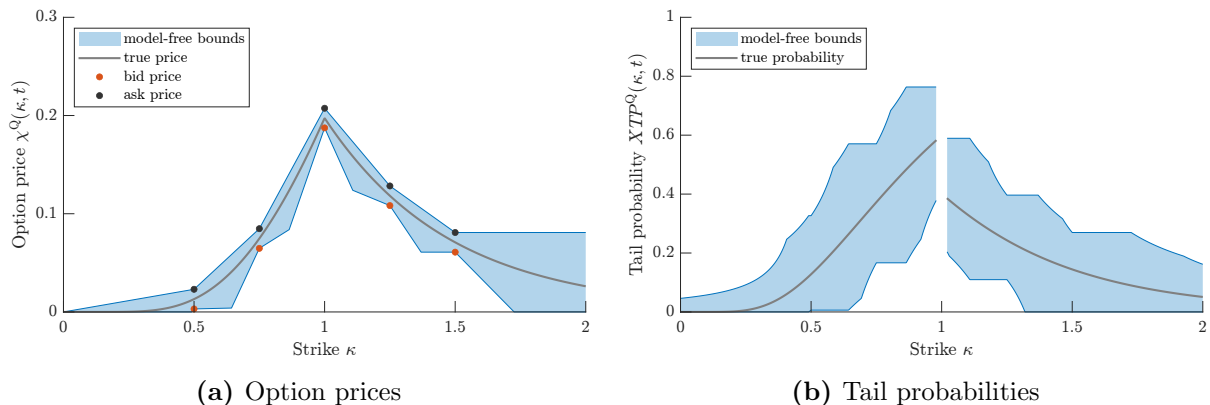


Figure 2.1: Illustrative option price and tail probability bounds

This figure illustrates the option price and tail probability bounds derived in section 3 in a simplified Black-Scholes economy. Options are written on the forward return R_t and quoted on a forward basis. True option prices $C(k, t)$ are given by the Black-Scholes formula with volatility 0.5. Bid prices $B(k, t) = C(k, t) - 0.01$ and ask prices $A(k, t) = C(k, t) + 0.01$ are observed for strike-maturity pairs $(k, t) \in \mathbb{K} \times \mathbb{T}$ on a strike grid $\mathbb{K} = \{0.5, 0.75, 1, 1.25, 1.5\}$ at the single maturity $\mathbb{T} = \{1\}$. Model-free bounds are determined at time horizon $\tau = 1$. Panel (a) plots the option price bounds in Proposition 3.1; it shows put prices for $\kappa < 1$ and call prices for $\kappa \geq 1$. Panel (b) plots the tail probability bounds in Proposition 3.3; it shows left-tail probabilities for $\kappa < 1$ and right-tail probabilities for $\kappa > 1$.

While the example Figure 2.1 only treats the single-period case, our theory provides explicit expressions for option price and tail probability bounds within a general multi-period, multi-maturity setting. The width of each of these bounds induces a natural measure of the associated valuation uncertainty, for which we suggest a decomposition to gauge the relative importance of the three notions of market incompleteness introduced above.

3 Model-free bounds and uncertainty quantification

This section derives model-free bounds and uncertainty measures for option prices and tail probabilities, where the latter can be interpreted as the (directional) slope of option prices in the strike dimension. The assumed market structure is formalized in section 3.1, which states

operational no-arbitrage conditions to ensure the martingale property of the underlying asset price. Given these preparations and exploiting the auxiliary results in Online Appendix OA, the problems of determining model-free option price and tail probability bounds can be readily cast as generalized moment problems. Explicit expressions for these bounds are provided in section 3.2 for option prices and in section 3.3 for tail probabilities, where also the induced uncertainty measures and their decompositions are constructed. Proofs of our main theoretical results are given in Online Appendix OB. Various further theoretical results are discussed in Online Appendix OC for option price bounds and in Online Appendix OD for probability bounds, while Online Appendix OE walks through several examples and Online Appendix OF provides details on the numerical implementation and performance.

3.1 Market setting

Take the continuous time domain $\mathcal{T} := [0, \infty)$. Our attention focuses on a risky asset with forward (gross) return process $(R_t : t \in \mathcal{T})$ given the deterministic initial value $R_0 = 1$. At each point in time t , the asset return $R_t := R_{0:t}$ between times 0 and t takes values in the state space $\Omega := [0, \infty)$. Throughout the exposition of our theory, we maintain a static perspective conditional on time 0.

By the Fundamental Theorem of Asset Pricing, absence of arbitrage is essentially equivalent to the existence of at least one "risk-neutral" probability measure relative to which the forward return process is a martingale (Delbaen and Schachermayer (1994), Harrison and Kreps (1979), and Harrison and Pliska (1981)). In a multi-period setting, absence of arbitrage imposes substantively rich conditions that a risk-neutral probability measure $\mathbb{Q}$ needs to satisfy in order to render the forward return process a martingale. As long as only marginal distributions are of interest, it suffices to impose constraints on the collection of marginal probability measures $\mathbb{Q}_{\mathcal{T}} := (\mathbb{Q}_t : t \in \mathcal{T})$ to assure the required martingale property in a particular form, without the need to model the full joint risk-neutral distribution. Here, each $\mathbb{Q}_t$ represents a time-0 conditional probability measure of R_t , for which time-0 conditional expectations are denoted by $E^{\mathbb{Q}_t}[f(R_t)]$ for a given function f . In order to enforce the absence of intertemporal arbitrage, our formulation specifically exploits a connection between martingales and the convex order of the collection $\mathbb{Q}_{\mathcal{T}}$, as rationalized in Online Appendix OA.1. This allows to tractably characterize the basic set of risk-neutral probability measures

$$\mathcal{Q}(\mathcal{T}) := \left\{ \mathbb{Q}_{\mathcal{T}} = (\mathbb{Q}_t : t \in \mathcal{T}) : \begin{array}{l} E^{\mathbb{Q}_t}[1] = 1 \text{ and } E^{\mathbb{Q}_t}[R_t] = 1 \text{ for all } t \in \mathcal{T}, \\ E^{\mathbb{Q}_t}[(R_t - k)^+] \leq E^{\mathbb{Q}_{t'}}[(R_{t'} - k)^+] \text{ for all } k \in \Omega \text{ and } t \leq t' \in \mathcal{T} \end{array} \right\},$$

where any collection of marginal probability measures $\mathbb{Q}_{\mathcal{T}}$ will henceforth simply be referred to as a probability measure. With this definition of $\mathcal{Q}(\mathcal{T})$, there is a one-to-one mapping between admissible risk-neutral probability measures and (statically) arbitrage-free option price surfaces, as formally defined and discussed in Online Appendix OA.2. This connection will be essential for us when determining explicit model-free bounds in what follows.

3.2 Bounds for option prices

Consider a plain-vanilla European option on R_t with specification (k, t, x) , where $k \in \Omega$ is the strike, $t \in \mathcal{T}$ is the time-to-maturity, and $x \in \{C, P\}$ is a flag that indicates whether the option is of call or put type. For each risk-neutral probability measure $\mathbb{Q}_\mathcal{T} \in \mathcal{Q}(\mathcal{T})$, the time-0 forward price of a European option on R_t associated to specification (k, t, x) satisfies

$$C^{\mathbb{Q}_\mathcal{T}}(k, t) = \mathbb{E}^{\mathbb{Q}_t}[(R_t - k)^+] , \quad P^{\mathbb{Q}_\mathcal{T}}(k, t) = \mathbb{E}^{\mathbb{Q}_t}[(k - R_t)^+] , \quad (3.1)$$

where $C^{\mathbb{Q}_\mathcal{T}}(k, t)$ denotes the call price and $P^{\mathbb{Q}_\mathcal{T}}(k, t)$ the put price. Both are related via the put-call parity relation $P^{\mathbb{Q}_\mathcal{T}}(k, t) = C^{\mathbb{Q}_\mathcal{T}}(k, t) + k - 1$.

Suppose that (forward) bid prices $B^x(k, t)$ and ask prices $A^x(k, t)$ of European options on R_t are observed for option specifications $(k, t, x) \in \mathbb{S}$. Without loss of generality, we assume that the set of observed option specifications takes the form of a regular grid, $\mathbb{S} := \mathbb{K} \times \mathbb{T} \times \{C, P\}$.⁵ At this point, $\mathbb{K} \subset \Omega$ is a set of strikes and $\mathbb{T} \subset \mathcal{T}$ is a set of maturities, on which no specific assumptions are imposed regarding their cardinality.⁶ In accordance with the general arbitrage theory in the presence of bid-ask spreads (e.g., Jouini and Kallal (1995) and Schachermayer (2004)), the observed bid and ask prices provide exogenous constraints on European option prices, yielding the set of admissible risk-neutral probability measures

$$\mathcal{O}(\mathcal{T}) := \left\{ \mathbb{Q}_\mathcal{T} \in \mathcal{Q}(\mathcal{T}) : \begin{array}{l} B^C(k, t) \leq C^{\mathbb{Q}_\mathcal{T}}(k, t) \leq A^C(k, t) \\ B^P(k, t) \leq P^{\mathbb{Q}_\mathcal{T}}(k, t) \leq A^P(k, t) \end{array} , (k, t) \in \mathbb{K} \times \mathbb{T} \right\} . \quad (3.2)$$

To ensure consistency of bid-ask constraints implied by call and put options, we further assume without loss of generality that the following put-call parity relations hold for bid and ask prices:

$$B^P(k, t) = B^C(k, t) + k - 1 , \quad A^P(k, t) = A^C(k, t) + k - 1 .$$

As long as put-call parity holds in this form, put and call constraints are redundant and we may equivalently to equation (3.2) define the set $\mathcal{O}(\mathcal{T})$ of admissible risk-neutral probability measures by

$$\mathcal{O}(\mathcal{T}) := \left\{ \mathbb{Q}_\mathcal{T} \in \mathcal{Q}(\mathcal{T}) : B^C(k, t) \leq C^{\mathbb{Q}_\mathcal{T}}(k, t) \leq A^C(k, t) , (k, t) \in \mathbb{K} \times \mathbb{T} \right\} . \quad (3.3)$$

The feasibility condition $\mathcal{O}(\mathcal{T}) \neq \emptyset$, which means that the system of observed bid-ask prices is arbitrage-free, is typically straightforward to verify (cf. Online Appendix OC.1).

⁵Missing data at some $(k, t) \in \mathbb{K} \times \mathbb{T}$ may be recorded for bid prices as $B^C(k, t) = (1 - k)^+$, $B^P(k, t) = (k - 1)^+$ and for ask prices as $A^C(k, t) = 1$, $A^P(k, t) = k$.

⁶Thereby, our main theoretical results apply, among others, to cases where $\mathbb{K} = \{k_1, \dots, k_n\}$ and $\mathbb{T} = \{t_1, \dots, t_m\}$ are finite sets, as for empirically observed option data, and to cases where $\mathbb{K} = [k_*, k^*]$ and $\mathbb{T} = [t_*, t^*]$ are intervals, which for instance arise when interpolating empirically observed data.

3.2.1 Model-free option price bounds

For any given option specification $(\kappa, \tau, \chi) \in \Omega \times \mathcal{T} \times \{C, P\}$, admissible risk-neutral probability measures in $\mathcal{O}(\mathcal{T})$ generally produce a non-trivial range of associated option prices $C^{\mathbb{Q}\tau}(\kappa, \tau)$ and $P^{\mathbb{Q}\tau}(\kappa, \tau)$. With the goal to characterize tight model-free bounds for these option prices, which do not rely on fundamental a priori restrictions on the set $\mathcal{Q}(\mathcal{T})$, we employ the machinery of generalized moment problems. As a result, this yields time-consistent option price bounds that encompass the information conveyed by the system of bid-ask prices across different strikes and incorporate the connection of maturities through the imposed convex order constraints, absent additional assumptions. Formally, we determine (optimal) lower and upper option price bounds from the problems

$$\inf_{\mathbb{Q}\tau \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}\tau \in \mathcal{O}(\mathcal{T})} C^{\mathbb{Q}\tau}(\kappa, \tau), \quad \inf_{\mathbb{Q}\tau \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}\tau \in \mathcal{O}(\mathcal{T})} P^{\mathbb{Q}\tau}(\kappa, \tau). \quad (3.4)$$

For option specification (κ, τ, χ) , the resulting lower bound (infimum) will be denoted by $\mathcal{L}^\chi(\kappa, \tau)$ and the upper bound (supremum) by $\mathcal{U}^\chi(\kappa, \tau)$. Any value between the lower and upper bound can be attained by some admissible probability measure $\mathbb{Q}\tau \in \mathcal{O}(\mathcal{T})$.⁷ Trivially, it holds that $\mathcal{L}^C(\kappa, 0) = \mathcal{U}^C(\kappa, 0) = (1 - \kappa)^+$ and $\mathcal{L}^P(\kappa, 0) = \mathcal{U}^P(\kappa, 0) = (\kappa - 1)^+$. Beyond that, it is straightforward to verify that call and put price bounds satisfy the put-call parity relations

$$\mathcal{L}^P(\kappa, \tau) = \mathcal{L}^C(\kappa, \tau) + \kappa - 1, \quad \mathcal{U}^P(\kappa, \tau) = \mathcal{U}^C(\kappa, \tau) + \kappa - 1. \quad (3.5)$$

Provided feasibility (i.e., $\mathcal{O}(\mathcal{T}) \neq \emptyset$), the following proposition provides explicit expressions for the bounds in problems (3.4); a proof of the proposition is provided in Online Appendix OB.1. The resulting expressions essentially take the form of linear interpolation and extrapolation rules involving the bid-ask constraints across different maturities. The upper bound $\mathcal{U}^\chi(\kappa, \tau)$ is entirely determined by the ask constraints, whereas the lower bound $\mathcal{L}^\chi(\kappa, \tau)$ depends on both bid and ask constraints. Economically, the lower and upper bounds correspond to the optimal choice out of several basic option strategies that sub- or super-replicate the option payoff (see Appendix A for a detailed discussion). The strategies employed for this purpose derive from the spread strategies considered by Cousot (2007) to characterize the absence of arbitrage.

Proposition 3.1 (Model-free option price bounds).

Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.4) for $k', k'' \in \mathbb{K}$ and $t', t'' \in \mathbb{T}$:

(i) The bounds for European calls with $\kappa \geq 0$, $\tau \geq 0$ are given by

$$\begin{aligned} \mathcal{U}^C(\kappa, \tau) = & 1 \wedge \{(1 - \kappa)^+ : \tau = 0\} \wedge \inf \{A^C(k', t') : k' \leq \kappa, t' \geq \tau\} \\ & \wedge \inf \left\{ \frac{k' - \kappa}{k'} + \frac{\kappa}{k'} A^C(k', t') : 0 < \kappa < k', t' \geq \tau \right\} \\ & \wedge \inf \left\{ \frac{k'' - \kappa}{k'' - k'} A^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') : k' \leq \kappa < k'', t', t'' \geq \tau \right\}, \end{aligned} \quad (3.6)$$

⁷Note that the bounds themselves may or may not be attainable, but in any case are approached arbitrarily closely by sequences of admissible probability measures.

$$\begin{aligned}
\mathcal{L}^C(\kappa, \tau) &= (1 - \kappa)^+ \vee \sup\{B^C(k', t') : \kappa \leq k', t' \leq \tau\} \\
&\vee \sup\left\{\frac{k' - \kappa}{k'} + \frac{\kappa}{k'} B^C(k', t') : 0 < k' \leq \kappa, t' \leq \tau\right\} \\
&\vee \sup\left\{\frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') : \kappa \leq k' < k'', t' \leq t'', \tau\right\} \\
&\vee \sup\left\{\frac{k'' - \kappa}{k'' - k'} A^C(k', t'') + \frac{\kappa - k'}{k'' - k'} B^C(k'', t') : k' < k'' \leq \kappa, t' \leq t'', \tau\right\}.
\end{aligned} \tag{3.7}$$

(ii) The bounds for European puts with $\kappa \geq 0, \tau \geq 0$ are given by

$$\begin{aligned}
\mathcal{U}^P(\kappa, \tau) &= \kappa \wedge \{(\kappa - 1)^+ : \tau = 0\} \wedge \inf\{A^P(k', t') + \kappa - k' : k' \leq \kappa, t' \geq \tau\} \\
&\wedge \inf\left\{\frac{\kappa}{k'} A^P(k', t') : 0 < \kappa < k', t' \geq \tau\right\} \\
&\wedge \inf\left\{\frac{k'' - \kappa}{k'' - k'} A^P(k', t') + \frac{\kappa - k'}{k'' - k'} A^P(k'', t'') : k' \leq \kappa < k'', t', t'' \geq \tau\right\},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathcal{L}^P(\kappa, \tau) &= (\kappa - 1)^+ \vee \sup\{B^P(k', t') + \kappa - k' : \kappa \leq k', t' \leq \tau\} \\
&\vee \sup\left\{\frac{\kappa}{k'} B^P(k', t') : 0 < k' \leq \kappa, t' \leq \tau\right\} \\
&\vee \sup\left\{\frac{k'' - \kappa}{k'' - k'} B^P(k', t') + \frac{\kappa - k'}{k'' - k'} A^P(k'', t'') : \kappa \leq k' < k'', t' \leq t'', \tau\right\} \\
&\vee \sup\left\{\frac{k'' - \kappa}{k'' - k'} A^P(k', t'') + \frac{\kappa - k'}{k'' - k'} B^P(k'', t') : k' < k'' \leq \kappa, t' \leq t'', \tau\right\}.
\end{aligned} \tag{3.9}$$

Some remarks are in order regarding the option price bounds in Proposition 3.1. Neither the lower bounds nor the upper bounds generally constitute arbitrage-free option price surfaces (cf. Lemma OA.2 for the relevant conditions). Specifically, except for particular cases, the lower bounds lack the required convexity in the strike dimension, whereas the upper bounds need not exhibit the required limit as $k \rightarrow \infty$.

Motivated by empirical market structures, the remainder of the discussion considers the practically most relevant case with a finite strike set $\mathbb{K} := \{k_* := k_1, \dots, k_n =: k^*\} \subset (0, \bar{K})$, where k_i are sorted in strictly increasing order, and a finite maturity set $\mathbb{T}$. In general, this case does not lead to any substantial simplification of the formulas in Proposition 3.1, apart from infima and suprema becoming ordinary minima and maxima, respectively. For practical purposes, these simplifications however enable the formulation of an efficient algorithm that determines the option price bound surfaces in piecewise-linear functional form for the entire strike-maturity domain $\Omega \times \mathcal{T}$, as presented in Online Appendix OF.1.⁸

In order to gain some intuitive insight into the fundamental structure of the option price bounds, we briefly discuss two illustrative examples in a simplified setting with a single maturity, for which particular cases are plotted in Figure 3.1. We use these examples to moreover concretely relate our results to the single-period option price bounds derived in Bertsimas and Popescu

⁸Accordingly, this explicit representation in functional form realizes substantial computational gains compared to existing numerical optimization approaches, which determine bounds separately for each strike-maturity pair.

(2002), effectively imposing the restrictions $\mathbb{T} = \{t\}$ and $\tau = t$. Compared to their results, Proposition 3.1 provides a significant generalization to a multi-period and multi-maturity setting, allowing for general maturity sets $\mathbb{T}$ and arbitrary target maturities τ . Further details and extensions of these examples are discussed in Online Appendix OE.

Example 3.1. Consider the case without bid-ask spreads and a single maturity $\mathbb{T} = \{t\}$. In particular, suppose that $B^C(k, t) = A^C(k, t) = C(k)$ for all $k \in \mathbb{K}$ at the traded maturity t , where $C(k)$ is a given function that represents arbitrage-free call prices. The absence of bid-ask spreads substantially simplifies the problem, as only immediately adjacent strikes are relevant in determining the option price bounds in Proposition 3.1. The resulting formulas for the bounds at $\tau = t$, which are stated in Online Appendix OE.2, correspond to those in Theorem 4 of Bertsimas and Popescu (2002).

An illustration of the option price bounds $\mathcal{L}^\chi(\kappa, t)$ and $\mathcal{U}^\chi(\kappa, t)$ in this case is shown in Figure 3.1(a). Essentially, option prices with maturity t are uniquely determined at any strike in $\mathbb{K}$, whereas strikes outside of $\mathbb{K}$ are generally exposed to non-zero valuation uncertainty, due to non-trivial ranges of possible option prices. These ranges are particularly wide in the tails. Further details and illustrations for the single-maturity case are provided in Online Appendix OE.2; extensions to multiple maturities are discussed in Online Appendices OE.5 and OE.8. ■

Example 3.2. Consider the case of non-zero bid-ask spreads and a single maturity $\mathbb{T} = \{t\}$. In this case, we have $B^C(k, t) < C(k) < A^C(k, t)$ for all $k \in \mathbb{K}$, where $C(k)$ likewise is some given function representing arbitrage-free call prices. Without additional structure on the observed bid-ask prices, Proposition 3.1 is needed in full generality after restricting to the special case $\mathbb{T} = \{t\}$, even when determining bounds at $\tau = t$. In particular, it generally does not suffice to merely consider the bid-ask prices at immediately adjacent strikes. Thereby and especially also due to the form of lower bounds, the option price bounds in Proposition 3.1 at $\tau = t$ differ structurally from those suggested by Theorem 7 in Bertsimas and Popescu (2002) for this case. Online Appendix OE.3 contains some further discussion of the differences and an illustrative comparison.

An illustration of the option price bounds $\mathcal{L}^\chi(\kappa, t)$ and $\mathcal{U}^\chi(\kappa, t)$ in the single-maturity case is shown in Figure 3.1(b). The presence of bid-ask spreads elevates the valuation uncertainty at all strikes, which now generally is non-zero even for an option with strike in $\mathbb{K}$ at the traded maturity t . Further details and illustrations for the single-maturity case are provided in Online Appendix OE.3; extensions to multiple maturities are discussed in Online Appendices OE.6 and OE.9. ■

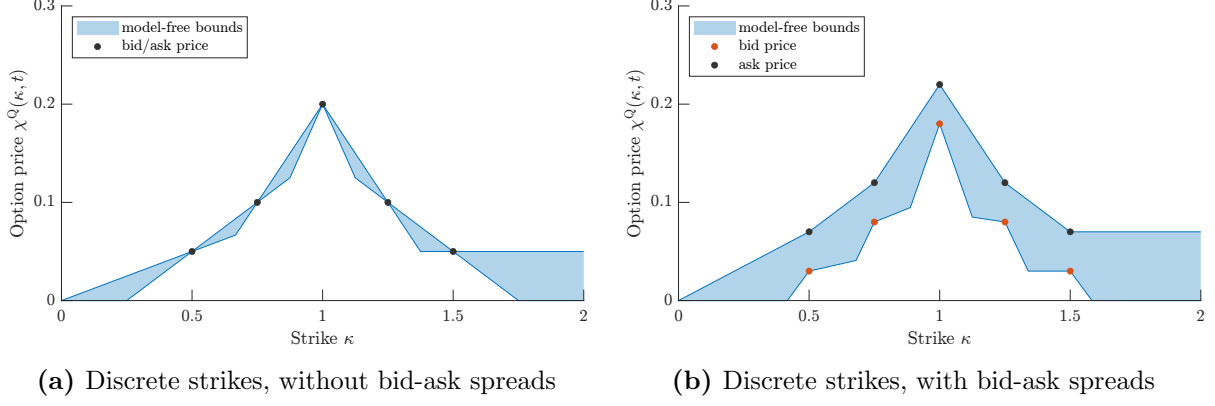


Figure 3.1: Illustrative option price bounds

This figure illustrates the option price bounds in Proposition 3.1 in a simplified setting. Option prices are observed on a strike grid $\mathbb{K} = \{0.5, 0.75, 1, 1.25, 1.5\}$ for the single maturity $\mathbb{T} = \{t\}$. Mid prices are specified as $C(0.5) = 0.55$, $C(0.75) = 0.35$, $C(1) = 0.2$, $C(1.25) = 0.1$, $C(1.5) = 0.05$. Bid and ask option prices are determined as $B^C(k, t) = C(k) - s/2$ and $A^C(k, t) = C(k) + s/2$ for all $k \in \mathbb{K}$ given a bid-ask spread s . In panel (a), we set $s = 0$; in panel (b), we set $s = 0.04$. The blue lines and shaded areas plot the option price bounds $\mathcal{L}^P(\kappa, t)$, $\mathcal{U}^P(\kappa, t)$ for $\kappa < 1$ and $\mathcal{L}^C(\kappa, t)$, $\mathcal{U}^C(\kappa, t)$ for $\kappa \geq 1$.

With a discrete strike grid $\mathbb{K}$, the expressions for the option price bounds in Proposition 3.1 directly imply a piecewise-linear dependence of the bounds on κ . Since the tail behavior of option prices for small and large κ plays an exposed role in the empirical literature (e.g., Bollerslev et al. (2015) and Gormsen and Jensen (2025)), it is worthwhile to further investigate the tail behavior of the option price bounds in Proposition 3.1, which is pursued in detail in Online Appendix OC.3. For each $\tau > 0$, one may construct a (bounded) minimum support $[z_*(\tau), z^*(\tau)]$, given by the explicit expressions in equations (OC.7) and (OC.8), where except for pathological cases $z_*(\tau) > 0$. In the left tail for $0 \leq \kappa \leq z_*(\tau)$ and $0 \leq \kappa \leq k_*$, respectively, we have for the put price bounds that

$$\mathcal{L}^P(\kappa, \tau) = 0, \quad \mathcal{U}^P(\kappa, \tau) = \frac{k_* - \kappa}{k_*} \mathcal{U}^P(k_*, \tau). \quad (3.10)$$

Likewise, in the right tail for $\kappa \geq z^*(\tau)$ and $\kappa \geq k^*$, respectively, the call price bounds behave as

$$\mathcal{L}^C(\kappa, \tau) = 0, \quad \mathcal{U}^C(\kappa, \tau) = \mathcal{U}^C(k^*, \tau). \quad (3.11)$$

Notably, both lower bounds eventually become zero, while the upper bounds for puts decay linearly to zero in the left tail and the upper bounds for calls stay constant in the right tail.

Relating to Lee (2004), we may consider the range of α for which $\mathbb{E}^{\mathbb{Q}_\tau}[R_\tau^\alpha]$ is bounded across all $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$. The possible tail behavior consistent with the option price bounds suggests that with a bounded range of observed strikes $\mathbb{K}$, $\mathcal{O}(\mathcal{T})$ does not convey any information to restrict the range of such α beyond the trivial range $[0, 1]$. In fact, by Lemma OC.7, option prices observed on a bounded strike grid (even when abstracting from bid-ask spreads and discrete strikes) are

consistent with (almost) any tail density.⁹ A thorough formal analysis of this observation and its severe consequences is performed in Bondarenko et al. (2024).

With a discrete maturity grid $\mathbb{T}$, the expressions for option price bounds in Proposition 3.1 are moreover piecewise-constant in τ . The bounds for any τ are entirely determined by the respective bounds at maturities in $\mathbb{T}$. Concretely, it holds that

$$\begin{aligned}\mathcal{L}^\chi(\kappa, \tau) &= \mathcal{L}^\chi(\kappa, \max\{t \in \mathbb{T} \cup \{0\} : t \leq \tau\}) , \\ \mathcal{U}^\chi(\kappa, \tau) &= \mathcal{U}^\chi(\kappa, \min\{t \in \mathbb{T} \cup \{0\} : t \geq \tau\}) .\end{aligned}\tag{3.12}$$

In particular, we obtain the trivial bounds $\mathcal{L}^C(\kappa, \tau) = (1 - \kappa)^+$, $\mathcal{L}^P(\kappa, \tau) = (\kappa - 1)^+$ for $\tau < \min \mathbb{T}$ as well as $\mathcal{U}^C(\kappa, \tau) = 1$, $\mathcal{U}^P(\kappa, \tau) = \kappa$ for $\tau > \max \mathbb{T}$. Even though it thereby suffices to characterize the option price bounds at individual maturities in $\mathbb{T}$, the bounds themselves generally differ from single-maturity bounds constructed subject to exclusively bid-ask constraints at the given maturity. Hence, the cross-maturity interactions of bid-ask prices in the formulas stated in Proposition 3.1 are indispensable in general. Nevertheless, precise relations between single-maturity and multi-maturity bounds are formally established in Online Appendix OC.4.

3.2.2 Maturity-interpolated option price bounds

As a consequence of equation (3.12), option price bounds can be expected to be relatively wide whenever $\tau \notin \mathbb{T}$. The discontinuous behavior of the bounds in the time dimension reflects that absence of arbitrage as such does not impose any smoothness for option prices in the maturity dimension (cf. condition (iv) in Lemma OA.2).

To contrast this behavior of the option price bounds, we use linear interpolation¹⁰ inbetween maturities in $\mathbb{T}$ to define the basic set of maturity-interpolated risk-neutral probability measures

$$\bar{\mathcal{Q}}(\mathcal{T}) := \left\{ \mathbb{Q}_\tau \in \mathcal{Q}(\mathcal{T}) : \mathbb{Q}_\tau = \frac{t'' - \tau}{t'' - t'} \mathbb{Q}_{t'} + \frac{\tau - t'}{t'' - t'} \mathbb{Q}_{t''} \text{ for all } \tau \notin \mathbb{T} \right\} ,$$

where $t' := \max\{t \in \mathbb{T} \cup \{0\} : t \leq \tau\}$ and $t'' := \min\{t \in \mathbb{T} : t > \tau\}$, employing constant extrapolation for any $\tau > \max \mathbb{T}$ (i.e., using the convention that $t'' := \infty$ in that case).

Denote by $\bar{\mathcal{O}}(\mathcal{T}) := \mathcal{O}(\mathcal{T}) \cap \bar{\mathcal{Q}}(\mathcal{T})$ the corresponding restriction of admissible risk-neutral probability measures to $\bar{\mathcal{Q}}(\mathcal{T})$. For a given option specification (κ, τ, χ) , problems (3.4) over admissible probability measures in $\bar{\mathcal{O}}(\mathcal{T})$ yield the lower bound $\bar{\mathcal{L}}^\chi(\kappa, \tau)$ and upper bound $\bar{\mathcal{U}}^\chi(\kappa, \tau)$. As $\bar{\mathcal{O}}(\mathcal{T}) \subset \mathcal{O}(\mathcal{T})$, we immediately have that $\bar{\mathcal{L}}^\chi(\kappa, \tau) \geq \mathcal{L}^\chi(\kappa, \tau)$ and $\bar{\mathcal{U}}^\chi(\kappa, \tau) \leq \mathcal{U}^\chi(\kappa, \tau)$, with equality when $\tau \in \mathbb{T}$. Moreover, the following proposition establishes that the option price bounds over $\bar{\mathcal{O}}(\mathcal{T})$ coincide with maturity-interpolation of the option price bounds in Proposition 3.1. A proof of the proposition is provided in Appendix OB.2.

⁹In particular, with regards to the power-law decay considered in Bollerslev and Todorov (2014) and Bollerslev et al. (2015), note that there generally exists a $\mathbb{Q}_\tau \in \mathcal{Q}(\mathcal{T})$ such that put prices $P^{\mathbb{Q}_\tau}(k, t)$ behave as $k^{1-\beta}$ in the left tail and call prices $C^{\mathbb{Q}_\tau}(k, t)$ behave as $k^{1-\gamma}$ in the right tail, for every $\beta < 0$ and $\gamma > 1$.

¹⁰Linear interpolation is straightforwardly compatible with the no-arbitrage conditions embedded through $\mathcal{Q}(\mathcal{T})$. Moreover, it is common in the literature in related contexts, such as the maturity-interpolation of option-implied moments. Online Appendix OC.5 discusses extensions to general monotonously increasing interpolation schemes.

Proposition 3.2 (Maturity-interpolated option price bounds).

Let $\mathbb{T} \subset \mathcal{T}$ be finite and $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.4) over $\bar{\mathcal{O}}(\mathcal{T})$:

$$\bar{\mathcal{U}}^\chi(\kappa, \tau) = \frac{t'' - \tau}{t'' - t'} \mathcal{U}^\chi(\kappa, t') + \frac{\tau - t'}{t'' - t'} \mathcal{U}^\chi(\kappa, t''), \quad (3.13)$$

$$\bar{\mathcal{L}}^\chi(\kappa, \tau) = \frac{t'' - \tau}{t'' - t'} \mathcal{L}^\chi(\kappa, t') + \frac{\tau - t'}{t'' - t'} \mathcal{L}^\chi(\kappa, t''), \quad (3.14)$$

where $t' := \max\{t \in \mathbb{T} \cup \{0\} : t \leq \tau\}$ and $t'' := \min\{t \in \mathbb{T} : t > \tau\}$.

3.2.3 Implied volatility bounds

In practice, it is common to quote option prices in terms of Black-Scholes implied volatilities. Formally, for an option specification (k, t, x) and an associated option price y , the corresponding Black-Scholes implied volatility $\Sigma^{BS}(k, t, x; y)$ is implicitly determined by

$$c = C^{BS}(k, t; \Sigma^{BS}(k, t, C; c)), \quad p = P^{BS}(k, t; \Sigma^{BS}(k, t, P; p)),$$

where c and p are some given call and put prices, respectively. If the put-call parity relation $p = c + k - 1$ holds, we particularly have the identity $\Sigma^{BS}(k, t, C; c) = \Sigma^{BS}(k, t, P; p)$.

Equivalently to problems (3.4), for a given option specification (κ, τ, χ) , we may determine (optimal) lower and upper implied volatility bounds from the problems

$$\inf_{\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})} \Sigma^{BS}(\kappa, \tau, C; C^{\mathbb{Q}_T}(\kappa, \tau)) = \inf_{\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})} \Sigma^{BS}(\kappa, \tau, P; P^{\mathbb{Q}_T}(\kappa, \tau)). \quad (3.15)$$

The resulting lower bound (infimum) is denoted by $\mathcal{L}^{IV}(\kappa, \tau)$ and the upper bound (supremum) by $\mathcal{U}^{IV}(\kappa, \tau)$. It is straightforward to verify that these implied volatility bounds simply correspond to the Black-Scholes implied volatilities of the option price bounds in Proposition 3.1. Hence, option price bounds and implied volatility bounds convey identical information and can easily be converted into one another.

Corollary 3.1 (Model-free implied volatility bounds).

Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.15):

$$\mathcal{U}^{IV}(\kappa, \tau) = \Sigma^{BS}(\kappa, \tau, C; \mathcal{U}^C(\kappa, \tau)) = \Sigma^{BS}(\kappa, \tau, P; \mathcal{U}^P(\kappa, \tau)), \quad (3.16)$$

$$\mathcal{L}^{IV}(\kappa, \tau) = \Sigma^{BS}(\kappa, \tau, C; \mathcal{L}^C(\kappa, \tau)) = \Sigma^{BS}(\kappa, \tau, P; \mathcal{L}^P(\kappa, \tau)). \quad (3.17)$$

An analogous result holds for problems (3.15) over admissible probability measures in $\bar{\mathcal{O}}(\mathcal{T})$. The resulting implied volatility bounds correspond to the Black-Scholes implied volatilities of the maturity-interpolated option price bounds in Proposition 3.2.

Corollary 3.2 (Maturity-interpolated implied volatility bounds).

Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.15) over $\bar{\mathcal{O}}(\mathcal{T})$:

$$\bar{\mathcal{U}}^{IV}(\kappa, \tau) = \Sigma^{BS}(\kappa, \tau, C; \bar{\mathcal{U}}^C(\kappa, \tau)) = \Sigma^{BS}(\kappa, \tau, P; \bar{\mathcal{U}}^P(\kappa, \tau)), \quad (3.18)$$

$$\bar{\mathcal{L}}^{IV}(\kappa, \tau) = \Sigma^{BS}(\kappa, \tau, C; \bar{\mathcal{L}}^C(\kappa, \tau)) = \Sigma^{BS}(\kappa, \tau, P; \bar{\mathcal{L}}^P(\kappa, \tau)). \quad (3.19)$$

3.3 Bounds for tail probabilities

European option prices are closely connected to the probability distribution of the underlying stock price. To exploit this connection, we consider for a given risk-neutral probability measure $\mathbb{Q}_\tau \in \mathcal{Q}(\mathcal{T})$ the left-tail and right-tail probabilities $LTP^{\mathbb{Q}_\tau}(k, t)$ and $RTP^{\mathbb{Q}_\tau}(k, t)$ associated to R_t at threshold level $k \in \Omega$ and time horizon $t \in \mathcal{T}$, defined by

$$LTP^{\mathbb{Q}_\tau}(k, t) := \mathbb{Q}_t(R_t \leq k), \quad RTP^{\mathbb{Q}_\tau}(k, t) := \mathbb{Q}_t(R_t \geq k). \quad (3.20)$$

Formally, $LTP^{\mathbb{Q}_\tau}(k, t) = \mathbb{E}^{\mathbb{Q}_t}[\mathbb{1}_{[0, k]}(R_t)] = \partial_{k+} P^{\mathbb{Q}_\tau}(k, t)$ and $RTP^{\mathbb{Q}_\tau}(k, t) = \mathbb{E}^{\mathbb{Q}_t}[\mathbb{1}_{[k, \infty)}(R_t)] = -\partial_{k-} C^{\mathbb{Q}_\tau}(k, t)$ are directly related to the (directional) slope of the option prices $P^{\mathbb{Q}_\tau}(k, t)$ and $C^{\mathbb{Q}_\tau}(k, t)$, where ∂_{k+} and ∂_{k-} denote the right-hand and left-hand derivative operator, respectively, with respect to the strike price.¹¹ Economically, $LTP^{\mathbb{Q}_\tau}(k, t)$ and $RTP^{\mathbb{Q}_\tau}(k, t)$ can also be interpreted as the time-0 forward prices of European digital options on R_t .

3.3.1 Model-free tail probability bounds

As for option prices, admissible risk-neutral probability measures in $\mathcal{O}(\mathcal{T})$ generally produce non-trivial ranges of tail probabilities $LTP^{\mathbb{Q}_\tau}(\kappa, \tau)$ and $RTP^{\mathbb{Q}_\tau}(\kappa, \tau)$ at any specification (κ, τ) . To quantify (optimal) lower and upper tail probability bounds, we consider the problems

$$\inf_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} LTP^{\mathbb{Q}_\tau}(\kappa, \tau), \quad \inf_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} RTP^{\mathbb{Q}_\tau}(\kappa, \tau). \quad (3.21)$$

The resulting lower bounds (infima) are denoted by $\mathcal{L}^{LTP}(\kappa, \tau)$, $\mathcal{L}^{RTP}(\kappa, \tau)$ and the upper bounds (suprema) by $\mathcal{U}^{LTP}(\kappa, \tau)$, $\mathcal{U}^{RTP}(\kappa, \tau)$. For convenience, we will occasionally use $\mathcal{L}^{XTP}(\kappa, \tau)$, $\mathcal{U}^{XTP}(\kappa, \tau)$ to refer generically to either of the left-tail and right-tail probability bounds. Any value between the lower and upper bounds can again be attained by admissible probability measures $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$. Trivially, we have that $\mathcal{L}^{LTP}(\kappa, 0) = \mathcal{U}^{LTP}(\kappa, 0) = \mathbb{1}_{[0, \kappa]}(1)$ and $\mathcal{L}^{RTP}(\kappa, 0) = \mathcal{U}^{RTP}(\kappa, 0) = \mathbb{1}_{[\kappa, \infty)}(1)$.

Provided feasibility (i.e., $\mathcal{O}(\mathcal{T}) \neq \emptyset$), the following proposition states explicit expressions for the tail probability bounds in problems (3.21) in terms of the option price bounds in Proposition 3.1; a proof of the proposition is provided in Online Appendix OB.3. Essentially, the

¹¹By definition, we set $\partial_{k+} f(k) = \lim_{h \downarrow 0} (f(k+h) - f(k))/h$ and $\partial_{k-} f(k) = \lim_{h \downarrow 0} (f(k) - f(k-h))/h$, where the limits exist if f is a convex function (e.g., Theorem 23.1 in Rockafellar (1970)). Note that for any $\mathbb{Q}_\tau \in \mathcal{Q}(\mathcal{T})$, both $C^{\mathbb{Q}_\tau}(\cdot, t)$ and $P^{\mathbb{Q}_\tau}(\cdot, t)$ are convex.

bounds are given as minima and maxima over directional finite-difference quotients involving lower and upper option price bounds at the maturity of interest.

Proposition 3.3 (Model-free tail probability bounds).

Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.21) for $k' \in \mathbb{K} \cup \{0, 1, \kappa\}$:

(i) The bounds for left-tail probabilities with $\kappa \geq 0$, $\tau \geq 0$ are given by

$$\mathcal{U}^{LTP}(\kappa, \tau) = 1 \wedge \inf \left\{ \frac{\mathcal{U}^P(k', \tau) - \mathcal{L}^P(\kappa, \tau)}{k' - \kappa} : \kappa < k' \right\}, \quad (3.22)$$

$$\mathcal{L}^{LTP}(\kappa, \tau) = \sup \left\{ \lim_{\eta \downarrow 0} \frac{\mathcal{L}^P(\kappa + \eta, \tau) - \mathcal{U}^P(k', \tau)}{\kappa + \eta - k'} : k' \leq \kappa \right\}. \quad (3.23)$$

(ii) The bounds for right-tail probabilities with $\kappa \geq 0$, $\tau \geq 0$ are given by

$$\mathcal{U}^{RTP}(\kappa, \tau) = \inf \left\{ -\frac{\mathcal{L}^C(\kappa, \tau) - \mathcal{U}^C(k', \tau)}{\kappa - k'} : k' < \kappa \right\}, \quad (3.24)$$

$$\mathcal{L}^{RTP}(\kappa, \tau) = 0 \vee \sup \left\{ -\lim_{\eta \downarrow 0} \frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' - (\kappa - \eta)} : \kappa \leq k' \right\}. \quad (3.25)$$

In the following discussion, we again focus on the practically most relevant case with a finite strike set $\mathbb{K} := \{k_* := k_1, \dots, k_n := k^*\} \subset (0, \bar{K})$ as well as a finite maturity set $\mathbb{T}$. In this case, the formulas in Proposition 3.3 likewise hold with infima and suprema replaced by minima and maxima, respectively, while the limits generally cannot be omitted. Online Appendix OF.1 provides an algorithm to compute the tail probability bounds in this setting.

To develop further intuition for the behavior of the tail probability bounds, we now return to the illustrative examples introduced in section 3.2, for which particular cases are plotted in Figure 3.2. Recall that both examples are situated in a setting where bid and ask prices are observed at a single maturity, while Proposition 3.3 applies much more broadly in multi-period settings. Further details and extensions of these examples are discussed in Online Appendix OE.

Example 3.1 (continued). For the case without bid-ask spreads and a single maturity $\mathbb{T} = \{t\}$, the tractable form of option price bounds allows to also derive closed-form expressions for the tail probability bounds in Proposition 3.3 at $\tau = t$. Appendix OE.2 states the relevant formulas in terms of directional finite differences of the observed option prices at immediately adjacent strikes in $\mathbb{K}$.

An illustration of the tail probability bounds $\mathcal{L}^{XTP}(\kappa, t)$ and $\mathcal{U}^{XTP}(\kappa, t)$ in this case is shown in Figure 3.2(a). Notably, tail probability bounds are substantially wider compared to the option price bounds. Moreover, we generally find non-zero uncertainty about tail probabilities at any strike, even those in $\mathbb{K}$, as the ranges of possible tail probabilities are non-trivial everywhere. Intuitively, this is because in order for tail probabilities to be pinned down, option prices need to be uniquely determined in a small neighborhood around a given strike, not only at isolated

strikes as for a finite set $\mathbb{K}$. Further details and illustrations for this case are provided in Online Appendix OE.2. ■

Example 3.2 (continued). For the case with bid-ask spreads and a single maturity $\mathbb{T} = \{t\}$, as for option price bounds, Proposition 3.3 is now needed in full generality after restricting to the special case $\mathbb{T} = \{t\}$ in order to obtain the respective tail probability bounds, even when considering $\tau = t$.

An illustration of the tail probability bounds $\mathcal{L}^{XTP}(\kappa, t)$ and $\mathcal{U}^{XTP}(\kappa, t)$ is shown in Figure 3.2(b). Naturally, the presence of bid-ask spreads increases the range of possible tail probabilities at any strike. The magnitude of the increase is substantial, especially also in the at-the-money region for $\kappa \approx 1$. Further details and illustrations for this case are provided in Online Appendix OE.3. ■

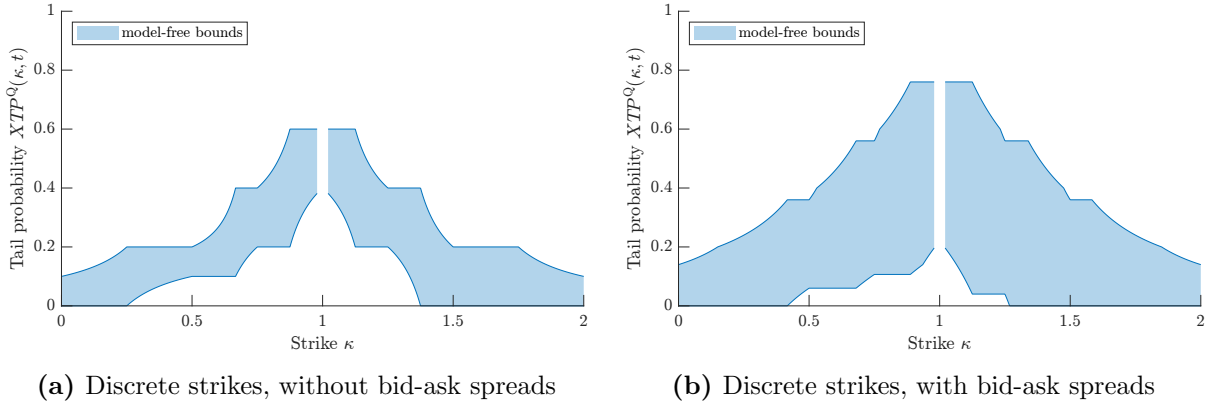


Figure 3.2: Illustrative tail probability bounds

This figure illustrates the tail probability bounds in Proposition 3.3 in a simplified setting. Option prices are observed on a strike grid $\mathbb{K} = \{0.5, 0.75, 1, 1.25, 1.5\}$ for the single maturity $\mathbb{T} = \{t\}$. Mid prices are specified as $C(0.5) = 0.55$, $C(0.75) = 0.35$, $C(1) = 0.2$, $C(1.25) = 0.1$, $C(1.5) = 0.05$. Bid and ask option prices are determined as $B^C(k, t) = C(k) - s/2$ and $A^C(k, t) = C(k) + s/2$ for all $k \in \mathbb{K}$ given a bid-ask spread s . In panel (a), we set $s = 0$; in panel (b), we set $s = 0.04$. The blue lines and shaded areas plot the tail probability bounds $\mathcal{L}^{LTP}(\kappa, t)$, $\mathcal{U}^{LTP}(\kappa, t)$ for $\kappa < 1$ and $\mathcal{L}^{RTP}(\kappa, t)$, $\mathcal{U}^{RTP}(\kappa, t)$ for $\kappa > 1$.

As for option price bounds, the tail behavior of the tail probability bounds in Proposition 3.3 for small and large strikes κ is of particular interest. To provide a concrete characterization of this behavior, for each $\tau > 0$, define the range $[y_*(\tau), y^*(\tau)]$ of non-trivial slopes of the upper option price bounds. In the left tail for small enough $0 \leq \kappa < z_*(\tau)$, as $\mathcal{L}^P(\kappa, \tau) = 0$, the left-tail probability bounds behave as

$$\mathcal{L}^{LTP}(\kappa, \tau) = 0, \quad \mathcal{U}^{LTP}(\kappa, \tau) = \frac{\mathcal{U}^P(y_*(\tau), \tau)}{y_*(\tau) - \kappa}, \quad (3.26)$$

where the lower bound equals zero in fact over the entire interval. Hence, in general, option prices observed on a bounded strike grid $\mathbb{K}$ cannot rule out risk-neutral probability measures with

positive point mass at the origin, as $\mathcal{U}^{LTP}(0, \tau) > 0$. In the right tail for large enough $\kappa > z^*(\tau)$, as $\mathcal{L}^C(\kappa, \tau) = 0$, the right-tail probability bounds exhibit the following behavior:

$$\mathcal{L}^{RTP}(\kappa, \tau) = 0, \quad \mathcal{U}^{RTP}(\kappa, \tau) = \frac{\mathcal{U}^C(y^*(\tau), \tau)}{\kappa - y^*(\tau)}. \quad (3.27)$$

In fact, the lower bound is attained as soon as $\mathcal{L}^C(\kappa, \tau) < \mathcal{U}^C(k^*, \tau)$, which is regularly the case already for smaller values of κ than $z^*(\tau)$.

3.3.2 Maturity-interpolated tail probability bounds

Due to equation (3.12), the tail probability bounds in Proposition 3.3 generally behave discontinuously in the maturity dimension. As for option price bounds, we thus consider analogues of problems (3.21) over admissible probability measures in $\bar{\mathcal{O}}(\mathcal{T})$. For a specification (κ, τ) , the resulting lower and upper bounds are denoted by $\bar{\mathcal{L}}^{XTP}(\kappa, \tau)$ and $\bar{\mathcal{U}}^{XTP}(\kappa, \tau)$, respectively. These bounds can be determined as in Proposition 3.3 when using the maturity-interpolated option price bounds from Proposition 3.2. By construction, it holds that $\bar{\mathcal{L}}^{XTP}(\kappa, \tau) \geq \mathcal{L}^{XTP}(\kappa, \tau)$ and $\bar{\mathcal{U}}^{XTP}(\kappa, \tau) \leq \mathcal{U}^{XTP}(\kappa, \tau)$. According to the following proposition, the tail probability bounds over $\bar{\mathcal{O}}(\mathcal{T})$ coincide with maturity-interpolation of the tail probability bounds in Proposition 3.3. A proof of the proposition is provided in Online Appendix OB.4.

Proposition 3.4 (Maturity-interpolated tail probability bounds).

Let $\mathbb{T} \subset \mathcal{T}$ be finite and $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.21) over $\bar{\mathcal{O}}(\mathcal{T})$:

$$\bar{\mathcal{U}}^{XTP}(\kappa, \tau) = \frac{t'' - \tau}{t'' - t'} \mathcal{U}^{XTP}(\kappa, t') + \frac{\tau - t'}{t'' - t'} \mathcal{U}^{XTP}(\kappa, t''), \quad (3.28)$$

$$\bar{\mathcal{L}}^{XTP}(\kappa, \tau) = \frac{t'' - \tau}{t'' - t'} \mathcal{L}^{XTP}(\kappa, t') + \frac{\tau - t'}{t'' - t'} \mathcal{L}^{XTP}(\kappa, t''), \quad (3.29)$$

where $t' := \max\{t \in \mathbb{T} \cup \{0\} : t \leq \tau\}$ and $t'' := \min\{t \in \mathbb{T} : t > \tau\}$.

3.4 Uncertainty quantification and decomposition

The levels of the various bounds derived in sections 3.2 and 3.3 are interesting in their own right. Beyond that, the width of each of these bounds (i.e., the distance between the lower and upper bound) yields a measure of the uncertainty about the respective quantity inherent in the observed system of bid-ask constraints (as captured by $\mathcal{O}(\mathcal{T})$). Accordingly, option price and tail probability bounds allow to quantify this uncertainty in different, complementary dimensions.

In fact, we may not only obtain measures of the aggregate uncertainty, but we can also construct a decomposition into the fundamental sources of option market incompleteness introduced in section 2. In order to prepare this decomposition, we consider the following hierarchy of uncertainty that can be applied to any of the bounds derived in sections 3.2 and 3.3:

- *Model-free uncertainty*: Use quoted bid and ask prices $B^C(k, t)$ and $A^C(k, t)$ for $(k, t) \in \mathbb{K} \times \mathbb{T}$ as constraints in $\mathcal{O}(\mathcal{T})$. Determine model-free bounds $\mathcal{L}(\kappa, \tau)$ and $\mathcal{U}(\kappa, \tau)$ over $\mathcal{O}(\mathcal{T})$ (e.g.,

from Propositions 3.1 and 3.3). Compute the uncertainty measure

$$\mathcal{D}(\kappa, \tau) := \mathcal{U}(\kappa, \tau) - \mathcal{L}(\kappa, \tau). \quad (3.30)$$

- *Maturity-interpolated uncertainty:* Use quoted bid and ask prices $B^C(k, t)$ and $A^C(k, t)$ for $(k, t) \in \mathbb{K} \times \mathbb{T}$ as constraints in $\mathcal{O}(\mathcal{T})$. Determine maturity-interpolated bounds $\bar{\mathcal{L}}(\kappa, \tau)$ and $\bar{\mathcal{U}}(\kappa, \tau)$ over $\bar{\mathcal{O}}(\mathcal{T})$ (e.g., from Propositions 3.2 and 3.4). Compute the uncertainty measure

$$\bar{\mathcal{D}}(\kappa, \tau) := \bar{\mathcal{U}}(\kappa, \tau) - \bar{\mathcal{L}}(\kappa, \tau). \quad (3.31)$$

- *Mid uncertainty:* Use hypothetical mid prices $C_M(k, t)$ for $(k, t) \in \mathbb{K} \times \mathbb{T}$ as constraints in $\mathcal{O}_M(\mathcal{T})$, assuming that $B^C(k, t) \leq C_M(k, t) \leq A^C(k, t)$.¹² Determine maturity-interpolated bounds $\bar{\mathcal{L}}_M(\kappa, \tau)$ and $\bar{\mathcal{U}}_M(\kappa, \tau)$ over $\bar{\mathcal{O}}_M(\mathcal{T})$ (e.g., from Propositions 3.2 and 3.4). Compute the uncertainty measure

$$\bar{\mathcal{D}}_M(\kappa, \tau) := \bar{\mathcal{U}}_M(\kappa, \tau) - \bar{\mathcal{L}}_M(\kappa, \tau). \quad (3.32)$$

By construction, these uncertainty measures are ordered as $\mathcal{D}(\kappa, \tau) \geq \bar{\mathcal{D}}(\kappa, \tau) \geq \bar{\mathcal{D}}_M(\kappa, \tau)$. This allows us to quantify contributions of the different sources of option market incompleteness through the following telescoping decomposition of the aggregate model-free uncertainty:

$$\mathcal{D}(\kappa, \tau) = \underbrace{\mathcal{D}(\kappa, \tau) - \bar{\mathcal{D}}(\kappa, \tau)}_{\text{maturity incompleteness}} + \underbrace{\bar{\mathcal{D}}(\kappa, \tau) - \bar{\mathcal{D}}_M(\kappa, \tau)}_{\text{quote incompleteness}} + \underbrace{\bar{\mathcal{D}}_M(\kappa, \tau)}_{\text{strike incompleteness}}. \quad (3.33)$$

The first component isolates effects due to maturity incompleteness, as it can be non-zero only if τ does not correspond to a traded maturity in $\mathbb{T}$. In that case, $\mathcal{D}(\kappa, \tau) - \bar{\mathcal{D}}(\kappa, \tau)$ quantifies the additional model-free uncertainty (due to weak no-arbitrage restrictions in the maturity dimension) relative to the interpolated uncertainty at the straddling traded maturities. After removing maturity effects through interpolation, the second component $\bar{\mathcal{D}}(\kappa, \tau) - \bar{\mathcal{D}}_M(\kappa, \tau)$ isolates the incremental uncertainty due to bid-ask quotes relative to the hypothetical mid prices. The third component $\bar{\mathcal{D}}_M(\kappa, \tau)$ measures the base uncertainty due to discrete strike grids that remains after controlling for maturity and bid-ask effects.

4 Empirical bounds for S&P 500 options

In this section, we determine option price and tail probability bounds using historical S&P 500 index option data. S&P 500 options rank among the most actively traded derivatives worldwide. They offer both a relatively long span of historical data and a rich cross section, with a wide range of strikes and maturities available on any given trading day. Accordingly, they permit

¹²In order to ensure feasibility, it is indispensable that $C_M(k, t)$ for $(k, t) \in \mathbb{K} \times \mathbb{T}$ are arbitrage-free. This will not automatically be assured when simply averaging observed bid and ask prices $B^C(k, t)$ and $A^C(k, t)$. To properly define an arbitrage-free set of mid prices, one may for instance use the so-called constrained convex regression approach suggested in Bondarenko (2003).

studying the evolution of option price and tail probability bounds over an extended period. The resulting bounds are expected to be relatively narrow compared to those in less liquid option markets. Hence, these bounds offer a plausible best-case estimate for the degree of uncertainty present in other equity option markets.

In addition, S&P 500 options are, by far, the most prominently featured options in empirical asset pricing research. We can therefore investigate consequences of our empirical results for a broad literature. In fact, bounds for option prices and tail probabilities are not only of interest as such, but are also directly relevant for any asset pricing applications using these quantities. In this regard, we specifically address their use in time series regressions, while explicitly accounting for the uncertainty within an incomplete option market setting.

We proceed in this section as follows. Section 4.1 describes the option data and its handling. Section 4.2 briefly addresses the existence of arbitrage opportunities, which need to be resolved prior to applying our methodology. Subsequently, sections 4.3 and 4.4 investigate the historical time series dynamics of option price and tail probability bounds for S&P 500 options, respectively. Finally, section 4.5 examines the implications of these bounds for empirical asset pricing applications, specifically time series regressions.

4.1 Data

Historical option data is obtained from OptionMetrics, available through WRDS. For the sample period from January 1996 to February 2023, we extract data for options written on the S&P 500 index on each available trading day, in particular closing bid and ask prices for put and call options, closing index levels, and interest rates; additionally, option-implied dividend yields are determined via put-call parity. Given the nature of our methodology, we apply only a bare minimum of data filters prior to conducting our analysis. In this regard, we only drop in-the-money put and call options, as their bid-ask ranges are regularly non-overlapping with those implied by out-of-the-money options, especially at extreme strikes.

Figure 4.1 provides an overview of the available strike-maturity pairs in the resulting sample. As evident from the plots, option data for the S&P 500 index has seen an increasing trend in the number of options per maturity band, the strike range in each maturity band, and the number of traded maturities, especially short-dated ones.

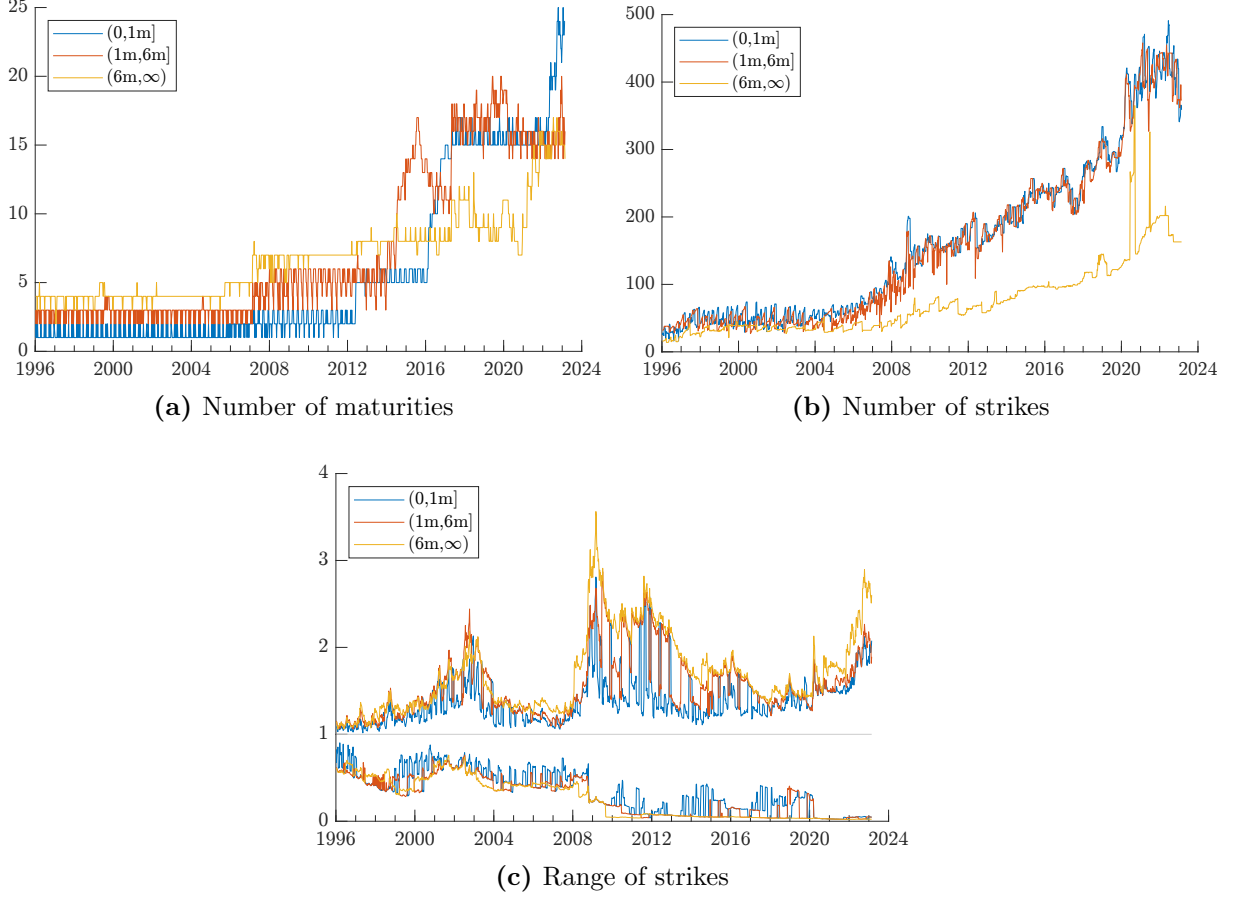


Figure 4.1: Option specifications

This figure shows summary statistics for the average number of maturities, average number of strikes per maturity, and average range of strikes per maturity, in three different maturity buckets.

4.2 Existence of arbitrage opportunities

As a by-product of our theoretical construction, we obtain a simple test for the existence of (static) arbitrage opportunities in empirical option markets when $\mathbb{K} \subset \Omega$ and $\mathbb{T} \subset \mathcal{T}$ are both finite.¹³ Following Davis and Hobson (2007), among others, we have that the existence of such arbitrage opportunities for a given set of bid-ask prices is equivalent to the infeasibility condition $\mathcal{O}(\mathcal{T}) = \emptyset$. To make use of our machinery to determine option price bounds, take $U^C(\kappa, \tau)$ to be the expression on the right-hand-side of equation (3.6). Note that $U^C(\kappa, \tau)$ as such may always be computed, even if the underlying problem is infeasible and thus the expression does not correspond to the upper bound in the call problem (3.4). It turns out that the values of $U^C(k, t)$ across all $(k, t) \in \mathbb{K} \times \mathbb{T}$ allow to directly verify the existence of (static) arbitrage opportunities, as a consequence of Lemma OC.2. Since each $U^C(k, t)$ may be determined in an efficient manner using the algorithm in Online Appendix OF.1, the following lemma yields a simplified version of

¹³More general no-arbitrage tests are provided by Lemmas OC.1 and OC.2 in Online Appendix OC.

the no-arbitrage test in Cousot (2007).

Lemma 4.1 (No-arbitrage test).

Let $\mathbb{K} \subset \Omega$ and $\mathbb{T} \subset \mathcal{T}$ both be finite. Then the following are equivalent:

- (i) $B^C(k, t)$ and $A^C(k, t)$ for $(k, t) \in \mathbb{K} \times \mathbb{T}$ are free of (static) arbitrage opportunities;
- (ii) $A^C(k, t) \geq (1 - k)^+$, $B^C(k, t) \leq U^C(k, t)$, and $B^C(k, t) < U^C(k', t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ and $k' < k$.

Table 4.1 reports summary statistics for arbitrage opportunities identified from Lemma 4.1.¹⁴ We find that no-arbitrage violations occur fairly regularly over the sample period, albeit typically just by individual options — with few exceptions. Violations are more frequent during the earlier part of the sample period and for (deep) out-of-the-money puts. Overall, the violations tend to be of relatively small magnitude. These findings complement the somewhat mixed empirical evidence under weaker notions of good deals or stochastic dominance in single-period settings (see Beare et al. (2025), Constantinides et al. (2020, 2009), and Post and Longarela (2021) as well as for very short maturities Almeida et al. (2025) and Chong and Todorov (2024)).

¹⁴We adopt the term "arbitrage opportunity" here for convenience. However, given the available data, we cannot make any statement about whether such an opportunity could have been profitably exploited in reality. Among others, such opportunities may artificially appear in the data due to price staleness or recording errors.

Table 4.1: Arbitrage opportunities

This table provides summary statistics for (static) arbitrage opportunities, aggregated into strike-maturity and year buckets. A no-arbitrage violation is defined as an option specification $(k, t) \in \mathbb{K} \times \mathbb{T}$ for which condition (ii) of Lemma 4.1 fails. The magnitude of a violation is defined as $(B^C(k, t) - U^C(k, t))^+$. For each strike-maturity and year bucket, the column *Days* counts the number of days for which at least one no-arbitrage violation has been recorded in that bucket; the column *Contrs* counts the average number of no-arbitrage violations, conditional on having at least one in that bucket; the column *Prices* reports the average magnitude of no-arbitrage violations, conditional on having at least one violation in that bucket.

Maturity	Strike	1996–2002			2003–2009			2010–2016			2017–2023		
		Days	Contrs	Prices	Days	Contrs	Prices	Days	Contrs	Prices	Days	Contrs	Prices
(0, 1/52]	(0, 0.90)	8	1.2	$7 \cdot 10^{-5}$	7	1.4	$5 \cdot 10^{-6}$	25	1.2	$4 \cdot 10^{-6}$	93	1.5	$1 \cdot 10^{-6}$
	[0.90, 0.97)	26	1.2	$9 \cdot 10^{-5}$	11	1.1	$1 \cdot 10^{-5}$	28	1.3	$8 \cdot 10^{-6}$	62	1.1	$1 \cdot 10^{-6}$
	[0.97, 1.03]	20	1.4	$4 \cdot 10^{-4}$	3	1.3	$6 \cdot 10^{-5}$	4	1.2	$2 \cdot 10^{-4}$	1	1.0	$4 \cdot 10^{-6}$
	(1.03, 1.10]	6	1.5	$2 \cdot 10^{-4}$	4	1.0	$1 \cdot 10^{-5}$	1	1.0	$1 \cdot 10^{-5}$	7	1.0	$2 \cdot 10^{-6}$
	(1.10, ∞)	0	—	—	1	1.0	$1 \cdot 10^{-4}$	2	1.0	$1 \cdot 10^{-5}$	2	1.0	$5 \cdot 10^{-6}$
(1/52, 1/12]	(0, 0.90)	53	1.4	$7 \cdot 10^{-5}$	21	1.7	$3 \cdot 10^{-5}$	73	1.8	$7 \cdot 10^{-6}$	48	1.6	$1 \cdot 10^{-6}$
	[0.90, 0.97)	41	1.6	$2 \cdot 10^{-4}$	3	1.0	$2 \cdot 10^{-5}$	4	6.8	$7 \cdot 10^{-5}$	2	7.0	$2 \cdot 10^{-4}$
	[0.97, 1.03]	44	1.4	$3 \cdot 10^{-4}$	6	2.7	$3 \cdot 10^{-4}$	2	2.5	$2 \cdot 10^{-4}$	1	15.0	$9 \cdot 10^{-4}$
	(1.03, 1.10]	34	1.5	$1 \cdot 10^{-4}$	5	1.2	$1 \cdot 10^{-5}$	5	3.2	$3 \cdot 10^{-5}$	6	2.0	$9 \cdot 10^{-5}$
	(1.10, ∞)	10	1.6	$1 \cdot 10^{-4}$	2	1.5	$9 \cdot 10^{-5}$	4	1.0	$8 \cdot 10^{-6}$	8	1.0	$7 \cdot 10^{-6}$
(1/12, 1/2]	(0, 0.90)	188	1.9	$2 \cdot 10^{-4}$	4	1.0	$8 \cdot 10^{-6}$	48	1.9	$7 \cdot 10^{-6}$	27	3.0	$5 \cdot 10^{-5}$
	[0.90, 0.97)	100	1.7	$5 \cdot 10^{-4}$	0	—	—	1	4.0	$2 \cdot 10^{-5}$	1	54.0	$2 \cdot 10^{-3}$
	[0.97, 1.03]	68	1.6	$7 \cdot 10^{-4}$	6	2.7	$2 \cdot 10^{-3}$	0	—	—	6	10.7	$5 \cdot 10^{-4}$
	(1.03, 1.10]	51	2.0	$7 \cdot 10^{-4}$	8	2.0	$1 \cdot 10^{-2}$	2	6.5	$6 \cdot 10^{-5}$	3	16.3	$5 \cdot 10^{-4}$
	(1.10, ∞)	29	2.2	$4 \cdot 10^{-4}$	1	1.0	$4 \cdot 10^{-6}$	2	1.0	$2 \cdot 10^{-5}$	7	1.4	$4 \cdot 10^{-6}$
(1/2, ∞)	(0, 0.90)	180	1.7	$7 \cdot 10^{-4}$	2	2.0	$1 \cdot 10^{-4}$	8	1.0	$2 \cdot 10^{-5}$	25	1.2	$1 \cdot 10^{-5}$
	[0.90, 0.97)	104	1.4	$1 \cdot 10^{-3}$	0	—	—	0	—	—	1	2.0	$6 \cdot 10^{-3}$
	[0.97, 1.03]	55	1.3	$1 \cdot 10^{-3}$	3	2.7	$2 \cdot 10^{-3}$	0	—	—	1	6.0	$2 \cdot 10^{-4}$
	(1.03, 1.10]	55	1.2	$2 \cdot 10^{-3}$	2	2.5	$4 \cdot 10^{-3}$	0	—	—	2	1.0	$4 \cdot 10^{-5}$
	(1.10, ∞)	37	1.7	$2 \cdot 10^{-3}$	0	—	—	0	—	—	0	—	—

For the remainder of our analysis, we require an option data set that is free of (static) arbitrage opportunities, so that $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Hence, we need to perform some further cleaning of the raw option data. For computing the summary statistics in Table 4.1, we blame any option that fails condition (ii) of Lemma 4.1 for a no-arbitrage violation. However, it can be (and, in fact, it occasionally happens in the sample) that a single ill-quoted option causes no-arbitrage violations in that sense for a series of options at adjacent strikes. Therefore, we do not simply remove all options that constitute no-arbitrage violations in the sense of Table 4.1, but we rather try to eliminate as few options as possible in order to restore the absence of arbitrage.

4.3 Bounds for option prices

To capture aspects of the historical evolution of observed option data, we select data on the first Wednesday in June of the years 2000, 2010, and 2020. On each of these selected sample dates, we determine model-free option price bounds according to Proposition 3.1, using observed bid and ask prices as constraints. For an easier interpretation of the resulting bounds, we convert option

price bounds into corresponding Black-Scholes implied volatility bounds as in Corollary 3.1. To gauge the magnitude of maturity effects, we also include the maturity-interpolated implied volatility bounds in Corollary 3.2, derived from the maturity-interpolated option price bounds in Proposition 3.2. Figure 4.2 displays the implied volatility bounds at 1-month maturity.

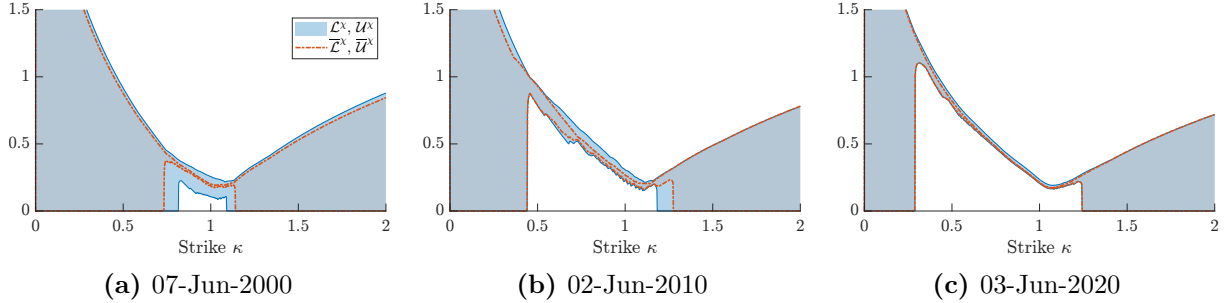


Figure 4.2: Cross section of implied volatility bounds ($\tau = 1\text{-month}$)

This figure shows implied volatility bounds $\mathcal{L}^{IV}(\kappa, \tau)$ and $\mathcal{U}^{IV}(\kappa, \tau)$ at maturity $\tau = 1\text{-month}$ on each sample date. The figure also shows the corresponding maturity-interpolated bounds $\bar{\mathcal{L}}^{IV}(\kappa, \tau)$ and $\bar{\mathcal{U}}^{IV}(\kappa, \tau)$.

Upper implied volatility bounds exhibit the typical smirk shape that is known from observed implied volatility curves. Lower implied volatility bounds follow this shape over a certain strike range around the at-the-money point, but ultimately become zero towards both boundaries of the state space. In this regard, it should be noted that the lower implied volatility bound is zero exactly when the corresponding lower option price bound is trivial (i.e., equal to the inner value). Since quoted bid prices may be zero, the lower bound may actually become trivial already within the traded strike range. Nevertheless, with the expansion of the range of traded strikes, particularly in the left tail, the lower bound tends to stay non-zero over wider strike intervals on the more recent sample dates. Implied volatility uncertainty is typically lowest for near-the-money options and increases towards the tails, with substantial uncertainty about the behavior of implied volatilities in the very deep tails.

The width of model-free implied volatility bounds within the traded strike range tends to decrease over time, but appears to be quite considerable especially on the earliest sample date. To put the width of the model-free bounds into perspective, it should be recalled that it effectively derives from lower and upper price bounds at the straddling traded maturities. At each traded maturity, the bounds are typically rather narrow, possibly over significantly wider strike intervals. Hence, the decreasing trend in the width of model-free bounds is primarily driven by more tightly spaced maturity bands. This effect is visualized in Figure 4.3, from which it becomes apparent how the introduction of additional maturities has led to a substantial reduction of maturity uncertainty, especially at shorter time horizons.

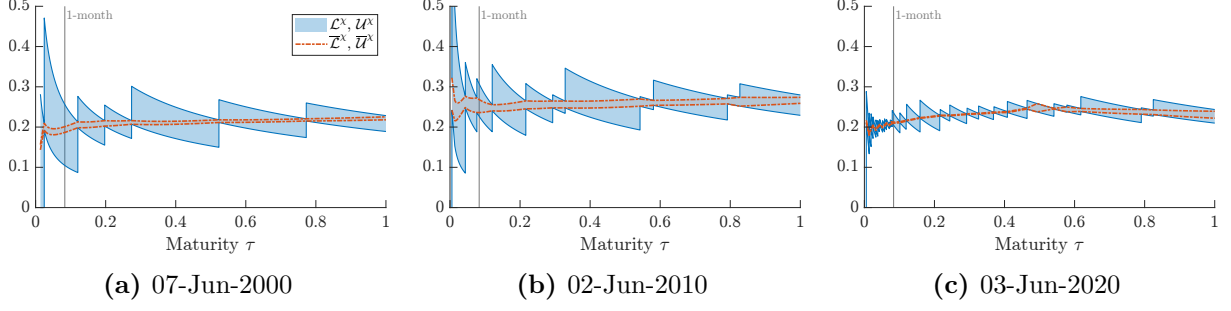


Figure 4.3: Term structure of implied volatility bounds ($\kappa = 1$)

This figure shows implied volatility bounds $\mathcal{L}^{IV}(\kappa, \tau)$ and $\mathcal{U}^{IV}(\kappa, \tau)$ at strike $\kappa = 1$ on each sample date. The figure also shows the corresponding maturity-interpolated bounds $\bar{\mathcal{L}}^{IV}(\kappa, \tau)$ and $\bar{\mathcal{U}}^{IV}(\kappa, \tau)$.

The preceding insights about the bounds on individual sample dates are helpful in order to understand their time series behavior over the entire sample period. To investigate this behavior, we determine the time series of the option price bounds in Proposition 3.1 at different log-moneyness levels $m = \log \kappa$. Subsequently, we again convert these price bounds into implied volatility bounds via Corollary 3.1 for better interpretability. We repeat this procedure for the maturity-interpolated option price bounds in Proposition 3.2, using the corresponding implied volatility bounds in Corollary 3.2. As common in the literature, we keep the focus of our analysis on the 1-month maturity, while choosing log-moneyness levels relative to some measure of volatility. Concretely, we take $m = \alpha \cdot v_t$ for some constant α , where we choose v_t to be either constant (proportional to the average at-the-money mid implied volatility) or time-varying (proportional to the daily at-the-money mid implied volatility). Figure 4.4 plots the time series of implied volatility bounds for out-of-the-money puts ($m < 0$), out-of-the-money calls ($m > 0$), and at-the-money options ($m = 0$). Due to their considerable day-to-day variations, we show monthly averages of the daily bounds.

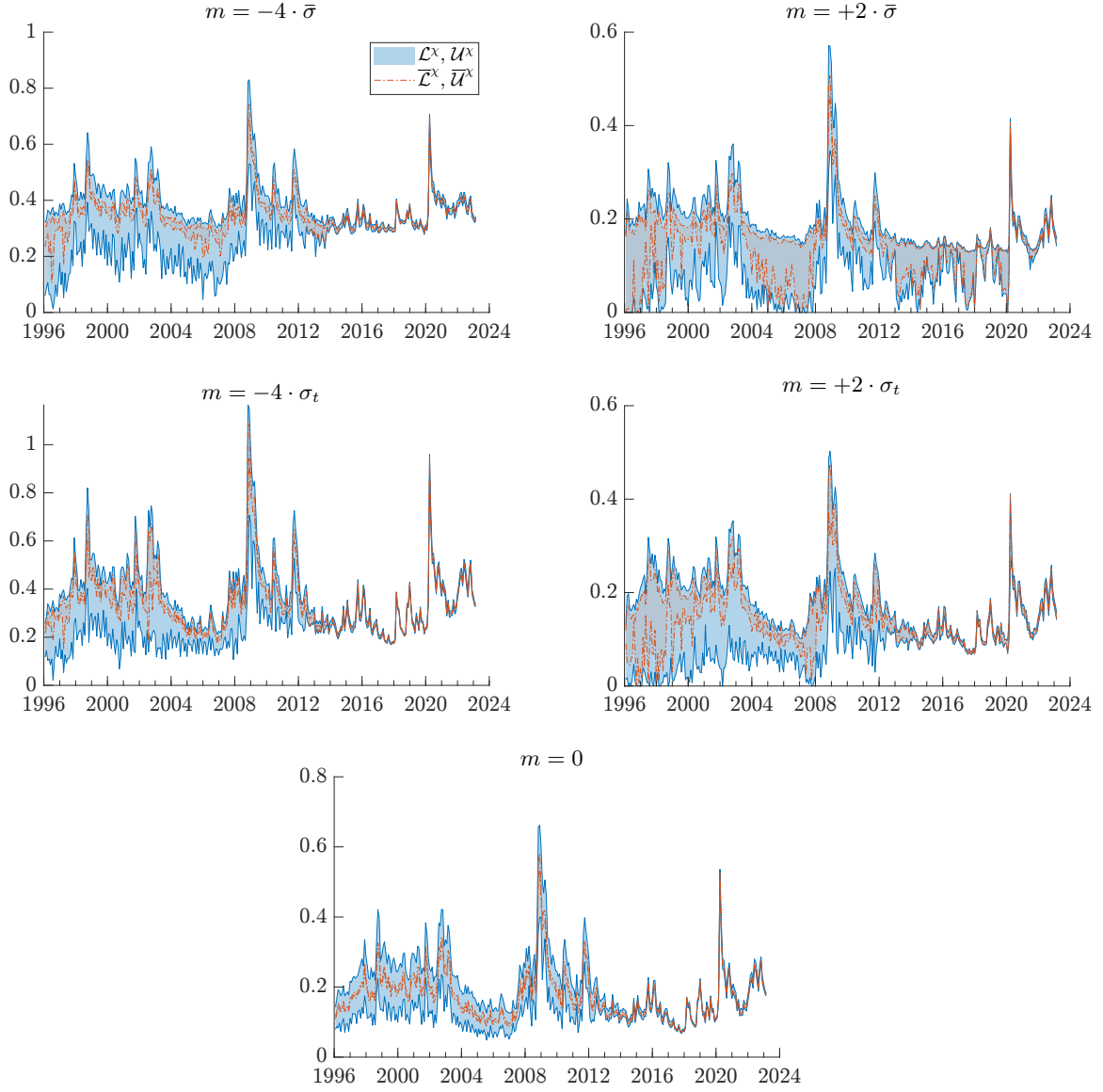


Figure 4.4: Time series of implied volatility bounds ($\tau = 1\text{-month}$)

This figure shows the time series of implied volatility bounds $\mathcal{L}^{IV}(e^m, \tau)$ and $\mathcal{U}^{IV}(e^m, \tau)$ at maturity $\tau = 1\text{-month}$. The figure also shows the corresponding maturity-interpolated bounds $\bar{\mathcal{L}}^{IV}(e^m, \tau)$ and $\bar{\mathcal{U}}^{IV}(e^m, \tau)$. The first row uses constant volatility scaling with $\bar{\sigma} = \sqrt{\tau} \bar{\text{IV}}^{ATM}(\tau)$; the second row uses time-varying volatility scaling with $\sigma_t = \sqrt{\tau} \text{IV}_t^{ATM}(\tau)$, where $\text{IV}_t^{ATM}(\tau)$ is the interpolated at-the-money mid implied volatility for maturity τ on date t and $\bar{\text{IV}}^{ATM}(\tau)$ is its sample average. The plotted bounds are monthly averages of daily bounds.

In Figure 4.4, we observe that both lower and upper bounds overall tend to move cyclically with the level of market volatility, as would be expected. However, especially in the earlier part of the sample period and for out-of-the-money calls, lower bounds may deviate depending on how extreme the respective moneyness level is relative to the traded strike range. When the lower bound hits zero near the boundary of the traded strike range, both model-free and maturity-interpolated bounds become wide. Otherwise, maturity-interpolated bounds are typically much narrower than model-free bounds, reflecting the important role played by maturity incompleteness.

Over the sample period, the width of implied volatility bounds decreases drastically. Except for out-of-the-money calls when m is constant, for which this development was delayed until after the Covid crash in 2020, implied volatility bounds — both model-free and maturity-interpolated — are substantially narrower especially since 2012. The historical evolution of the bounds aligns with the surge of traded maturities starting in 2012 and the gradual increase of traded strike ranges throughout the entire sample period. This suggests a downward trend in the inherent option price uncertainty, whose structural behavior will be investigated in section 5.1.

4.4 Bounds for tail probabilities

Analogous to the starting point for option prices, our empirical analysis of tail probabilities begins with a comparison across three selected sample dates. On each of these dates, we determine the model-free bounds for tail probabilities in Proposition 3.3, as well as the maturity-interpolated bounds in Proposition 3.4, using the respective option price bounds in section 4.3. Figure 4.5 shows the resulting left-tail and right-tail probability bounds at 1-month maturity.

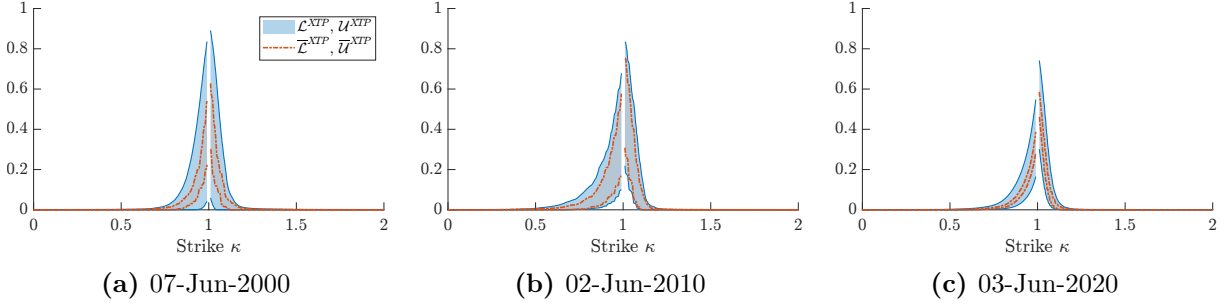


Figure 4.5: Cross section of tail probability bounds ($\tau = 1\text{-month}$)

This figure shows tail probability bounds $\mathcal{L}^{XTP}(\kappa, \tau)$ and $\mathcal{U}^{XTP}(\kappa, \tau)$ at maturity $\tau = 1\text{-month}$ on each sample date. The figure also shows the corresponding maturity-interpolated bounds $\bar{\mathcal{L}}^{XTP}(\kappa, \tau)$ and $\bar{\mathcal{U}}^{XTP}(\kappa, \tau)$. In each panel on the particular date, the left part of the plot ($\kappa < 1$) shows left-tail probabilities; the right part of the plot ($\kappa > 1$) shows right-tail probabilities.

Due to the relatively short time-to-maturity, tail probability bounds are quite narrowly centered around the at-the-money point. Notably, model-free bounds are rather wide in the center of the strike range, especially on the earliest sample date. The model-free lower bound may become zero already quite close to the at-the-money point, where it should be recalled that this happens as soon as the lower price bound falls below the upper price bounds at more extreme strikes. This is particularly likely when the straddling traded maturities are farther apart, as in the earlier part of the sample. Maturity-interpolated bounds are typically much narrower and positive over wider strike intervals, but irrespective of this feature larger uncertainty in the near-the-money region around $\kappa = 1$ compared to the tails. The decreasing width of option price bounds over time immediately translates into a decreasing width of tail probability bounds.

To analyze the behavior of tail probability bounds over the entire sample period, we compute the time series of model-free tail probability bounds in Proposition 3.3 and their maturity-

interpolated counterparts in Proposition 3.4. As for option price bounds, our focus remains on the 1-month maturity. We again consider different log-moneyness levels $m = \alpha \cdot v_t$, one in the left tail ($\alpha = -4$) and one in the right tail ($\alpha = +2$).¹⁵ Moreover, we explicitly contrast the cases with constant and time-varying m , as both are considered in the literature¹⁶ and exhibit very different structural dynamics. Figure 4.6 plots the time series of left-tail ($m < 0$) and right-tail ($m > 0$) probability bounds, again in the form of monthly averages of the daily bounds.

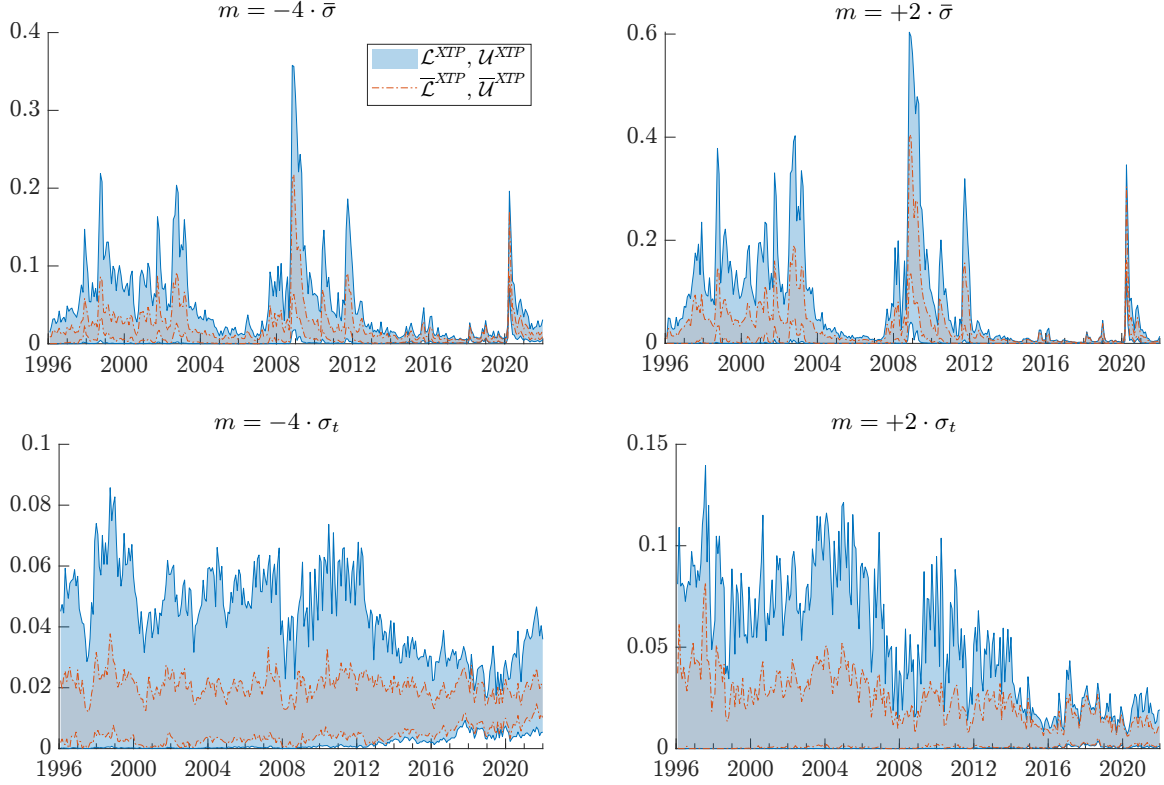


Figure 4.6: Time series of tail probability bounds ($\tau = 1$ -month)

This figure shows the time series of tail probability bounds $\mathcal{L}^{XTP}(e^m, \tau)$ and $\mathcal{U}^{XTP}(e^m, \tau)$ at maturity $\tau = 1$ -month. The figure also shows the corresponding maturity-interpolated bounds $\bar{\mathcal{L}}^{XTP}(e^m, \tau)$ and $\bar{\mathcal{U}}^{XTP}(e^m, \tau)$. The left column ($m < 0$) shows left-tail probabilities; the right column ($m > 0$) shows right-tail probabilities. The top row uses constant volatility scaling with $\bar{\sigma} = \sqrt{\tau} \bar{\text{IV}}^{ATM}(\tau)$; the bottom row uses time-varying volatility scaling with $\sigma_t = \sqrt{\tau} \text{IV}_t^{ATM}(\tau)$, where $\text{IV}_t^{ATM}(\tau)$ is the interpolated at-the-money mid implied volatility for maturity τ on date t and $\bar{\text{IV}}^{ATM}(\tau)$ is its sample average. The plotted bounds are monthly averages of daily bounds.

The fundamentally different behavior of tail probability bounds for constant and time-varying m is immediately apparent from Figure 4.6. In the constant case, the bounds move cyclically with the level of market volatility. In the time-varying case, the bounds do not follow a clear cyclicity pattern, but display some weak signs of counter-cyclical behavior relative to market volatility.

¹⁵As a benchmark, a Black-Scholes model with correctly specified implied volatility yields left-tail probability 0.0032% and right-tail probability 2.28%.

¹⁶For instance, Martin and Shi (2025) effectively consider a constant m to infer option-implied crash probabilities. In contrast, Gormsen and Jensen (2025), following Bollerslev et al. (2015), employ a time-varying m to investigate time series properties of tail probabilities.

For model-free bounds in both cases, the patterns are primarily driven by upper bounds, while lower bounds for the most part of the sample stay at or very close to zero. Maturity-interpolated bounds are much narrower, with lower bounds that at least in the left tail move somewhat farther away from zero. This again emphasizes the important role of the inherent uncertainty due to maturity incompleteness. Yet, a substantial amount of uncertainty remains after removing this component, as maturity-interpolated bounds are still rather wide in absolute terms.

To appreciate the economic significance of the typical magnitude of these bounds, for instance, observe for the case of time-varying m that over large parts of the sample period, left-tail and right-tail probabilities could have taken any value between close to 0% and 2% (within maturity-interpolated bounds), or close to 0% and 4% (within model-free bounds). The bounds are particularly wide during the earlier part of the sample period and become narrower in the more recent part, reflecting the analogous behavior of option price (implied volatility) bounds observed previously. Accordingly, this suggests declining tail probability uncertainty over the sample period, alongside the option price uncertainty. A confirmation of this trend as well as further analyses of the tail probability uncertainty are deferred to section 5.2.

Overall, the width and shape of the tail probability bounds suggests that observed option data may accommodate many forms of time series behavior of "true" tail probabilities. Over vast parts of the sample, the time series behavior may even be (almost) arbitrary. Given that this observation can already be made for S&P 500 index options, less developed option markets can be expected to be affected to an even larger degree. Therefore, the level of uncertainty embedded in observed option prices significantly interferes with attempts to establish concrete time series properties or dependencies with other economic variables.

4.5 Empirical asset pricing applications

One of the most prominent uses of option-based information in the empirical asset pricing literature is in time series regressions. For instance, option-implied probabilities or moments are employed within various prediction exercises (see Christoffersen et al. (2013) and, more recently, Martin (2025) for comprehensive reviews), or to investigate time series relations with other economic variables (e.g., recently Gormsen and Jensen (2025)). Naturally, we keep our focus here on applications involving option prices (implied volatilities) and tail probabilities. Other applications that rely on more general moments are beyond the scope of this paper, but are addressed in Bondarenko et al. (2024) and Dillschneider (2025).

The key premise underlying any such applications is that observed option prices contain useful, forward-looking information that can be extracted at high precision. Our empirical results in the preceding sections 4.3 and 4.4 challenge the final part of this premise. However, at this point, it is not yet clear to which degree uncertainty in option-based regression inputs propagates to uncertainty of regression outputs, such as coefficient estimates. While the typically employed econometric methodologies neglect this uncertainty entirely, our framework can be extended to directly quantify its effects on the outcomes of time series regressions.

To study the implications of our framework within a time series setting, we need to be explicit about the time dependence of the construction. Concretely, we suppose that at each point in time $t = 1, \dots, T$, one observes bid-ask prices of options written on the τ -period forward gross return $R_{t:t+\tau}$ between t and $t + \tau$, having different strikes k and times-to-maturity τ . The observed option prices at time t yield an associated set of admissible risk-neutral probability measures $\mathcal{O}_t(\mathcal{T})$ for a common $\mathcal{T}$. Each admissible choice $\mathbb{Q}_{\mathcal{T},t} = (\mathbb{Q}_{\theta,t} : \theta \in \mathcal{T}) \in \mathcal{O}_t(\mathcal{T})$ produces a quantity $X_t^{\mathbb{Q}_{\mathcal{T},t}} := \mathbb{E}^{\mathbb{Q}_{\mathcal{T},t}}[f(R_{t:t+\tau})]$ for a given payoff function f , such as an option price as in section 3.2 or a tail probability as in section 3.3, or transformations thereof. These $X_t^{\mathbb{Q}_{\mathcal{T},t}}$ are then used to explain an (observable) quantity Y_t in time series regressions of the form

$$Y_t = \alpha + \beta X_t^{\mathbb{Q}_{\mathcal{T},t}} + \varepsilon_t. \quad (4.1)$$

Similarly, an (observable) quantity Y_t may be used to explain $X_t^{\mathbb{Q}_{\mathcal{T},t}}$ in time series regressions of the form

$$X_t^{\mathbb{Q}_{\mathcal{T},t}} = \alpha + \beta Y_t + \varepsilon_t. \quad (4.2)$$

As the admissible probability measures are not uniquely pinned down, any outcome of regression (4.1) will implicitly be a function of the chosen collection $\mathbb{Q}_{\mathcal{T},1:T} := (\mathbb{Q}_{\mathcal{T},t} : t = 1, \dots, T)$. To explicitly represent this dependence, we thus denote the least-squares estimate of β by $\hat{\beta}^{\mathbb{Q}_{\mathcal{T},1:T}}$ and the resulting coefficient of determination by $R^{2,\mathbb{Q}_{\mathcal{T},1:T}}$. Bondarenko et al. (2024) show within a similar setting that the outcomes of regressions (4.1) and (4.2) become entirely uninformative in the extreme case where f cannot be measured robustly (i.e., has infinite bounds). In the following lemma, we strengthen this result further to cases where the bounds are finite, but still dominate in a certain sense the time series behavior. The implications are not entirely symmetric between regressions (4.1) and (4.2). It should further be noted that any value between the stated bounds can be realized by some admissible $\mathbb{Q}_{\mathcal{T},1:T}$.

Lemma 4.2 (Time series regressions).

Suppose that $[L_t, U_t] \supset [L^, U^*]$ for all $t = 1, \dots, T$ and some $L^* < U^*$, where $L_t := \mathcal{L}[X_t^{\mathbb{Q}_{\mathcal{T},t}]$ and $U_t := \mathcal{U}[X_t^{\mathbb{Q}_{\mathcal{T},t}]$ denote the bounds for $X_t^{\mathbb{Q}_{\mathcal{T},t}}$ obtained at time t . Then the following holds in regression (4.1):*

(i) *The estimate of β has bounds $\mathcal{L}[\hat{\beta}^{\mathbb{Q}_{\mathcal{T},1:T}}] = -\infty$ and $\mathcal{U}[\hat{\beta}^{\mathbb{Q}_{\mathcal{T},1:T}}] = +\infty$.*

(ii) *The R^2 has bounds $\mathcal{L}[R^{2,\mathbb{Q}_{\mathcal{T},1:T}}] = 0$ and $\mathcal{U}[R^{2,\mathbb{Q}_{\mathcal{T},1:T}}] = 1$.*

Moreover, the following holds in regression (4.2):

(iii) *The R^2 has bounds $\mathcal{L}[R^{2,\mathbb{Q}_{\mathcal{T},1:T}}] = 0$ and $\mathcal{U}[R^{2,\mathbb{Q}_{\mathcal{T},1:T}}] = 1$.*

Lemma 4.2 represents an extreme case within the time series regression framework. However, it emphasizes that even bounded uncertainty about $X_t^{\mathbb{Q}_{\mathcal{T},t}}$ may have substantial implications for regression outcomes. In less clear-cut situations, uncertainty about $X_t^{\mathbb{Q}_{\mathcal{T},t}}$ will still be reflected

in non-trivial, possibly wide bounds for $\hat{\beta}^{\mathbb{Q}_{\tau,1:T}}$ and $R^{2,\mathbb{Q}_{\tau,1:T}}$, which need to be determined numerically.

Effectively, the condition we impose in Lemma 4.2 requires that lower and upper bounds of $X_t^{\mathbb{Q}_{\tau,t}}$ span a constant corridor across time. The underlying intuition is that within such a corridor, $X_t^{\mathbb{Q}_{\tau,t}}$ is free to express *any* form of time series dynamic (up to appropriate scaling). In contrast, when lower and upper bounds already prescribe a strong time series behavior, the range of possible patterns that $X_t^{\mathbb{Q}_{\tau,t}}$ can follow is more limited. Hence, in regressions, not only the absolute width of the bounds matters, but their entire time series behavior.

The corridor condition may appear somewhat artificial at first sight, but in fact Figure 4.6 suggests that tail probability bounds, when using time-varying moneyiness levels, fit this condition quite closely. On the other hand, the case with constant moneyiness levels features bounds that superimpose strong time series movements. Consequently, we would expect these two cases to yield markedly different implications within a regression setting. We will illustrate these differences in several empirical asset pricing applications

5 Empirical uncertainty in S&P 500 options

As a by-product of our investigation of option price and tail probability bounds, the analysis of the preceding section 4 offers an indication of the magnitude of uncertainty in the S&P 500 index option market. In the present section, we now shift the focus entirely to this uncertainty.

Option price and tail probability uncertainty, which naturally arise as differences of their bounds studied in section 4, are themselves of first-order economic interest. Each such uncertainty represents the maximum disagreement about the given quantity (option price or tail probability with a certain specification) that is viable in an arbitrage-free option market. Accordingly, these option-implied uncertainty measures provide real-time, forward-looking information about the state of the option market, and potentially also the broader state of the economy.

In order to assess their information content, sections 5.1 and 5.2 provide a comprehensive analysis of the dynamic properties of the option price and tail probability uncertainty inherent in S&P 500 index options. Subsequently, section 5.3 investigates the relation of our uncertainty measures to variables that represent the state of the economy.

5.1 Uncertainty of option prices

In order to investigate the dynamic properties of option price uncertainty more closely, Figure 5.1 plots uncertainty measures derived from the implied volatility bounds at 1-month maturity in Figure 4.4. The computation of these measures follows section 3.4.

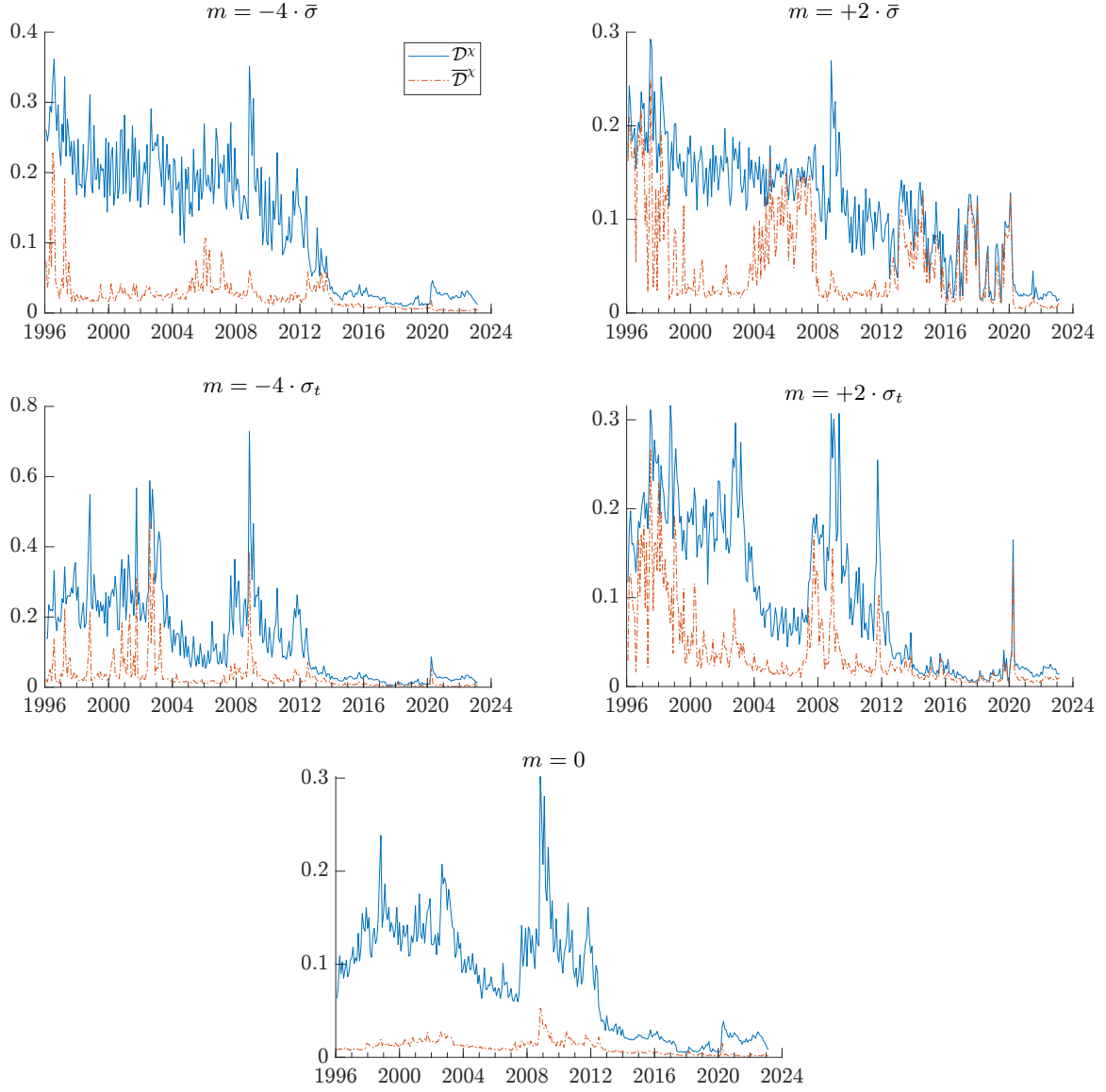


Figure 5.1: Time series of implied volatility uncertainty ($\tau = 1$ -month)

This figure shows the time series of implied volatility uncertainty $\mathcal{D}^{IV}(e^m, \tau)$ at maturity $\tau = 1$ -month. The figure also shows the corresponding maturity-interpolated uncertainty $\bar{\mathcal{D}}^{IV}(e^m, \tau)$ and mid uncertainty $\bar{\mathcal{D}}_M^{IV}(e^m, \tau)$. The plots use constant volatility scaling with $\bar{\sigma} = \sqrt{\tau} \bar{\text{IV}}^{ATM}(\tau)$, where $\text{IV}_t^{ATM}(\tau)$ is the interpolated at-the-money mid implied volatility for maturity τ on date t and $\bar{\text{IV}}^{ATM}(\tau)$ is its sample average. The plotted uncertainties are monthly averages of daily uncertainties.

For at-the-money options, Figure 5.1 reveals some cyclical co-movement with market volatility. Model-free uncertainty typically remains at an elevated level of 10% or more prior to 2012 and declines sharply afterwards. Maturity-interpolated uncertainty is much lower throughout, but also gradually decreases further from 2012 onward. Similar time series patterns can also be recognized for implied volatility uncertainty at the other moneyness levels, while also tail-specific effects come into play. These produce heterogeneous behavior of implied volatility uncertainty for constant and time-varying m . In particular, with regards to model-free uncertainty, we observe a

more steady decrease over time in the case with constant m , whereas a stronger co-movement with market volatility in the case with time-varying m . Maturity-interpolated uncertainty does not exhibit an equally pronounced downward trend before 2012, but tends to gradually decrease afterwards. In this regard, recall from Figure 4.2 that model-free and maturity-interpolated uncertainty do not need to move in lockstep, as the latter may be better behaved over wider strike intervals. Apart from periods of high market volatility, maturity-interpolated uncertainty spikes in particular when m is too large compared to the traded strike range, consistent with the observations made in Figure 4.4. During these episodes, strike (tail) incompleteness becomes the dominant source of uncertainty, whereas otherwise maturity incompleteness generally is the primary source.

5.2 Uncertainty of tail probabilities

Figure 5.2 plots the time series of tail probability uncertainty, derived from the tail probability bounds at 1-month maturity in Figure 4.6. These measures are again computed as in section 3.4.

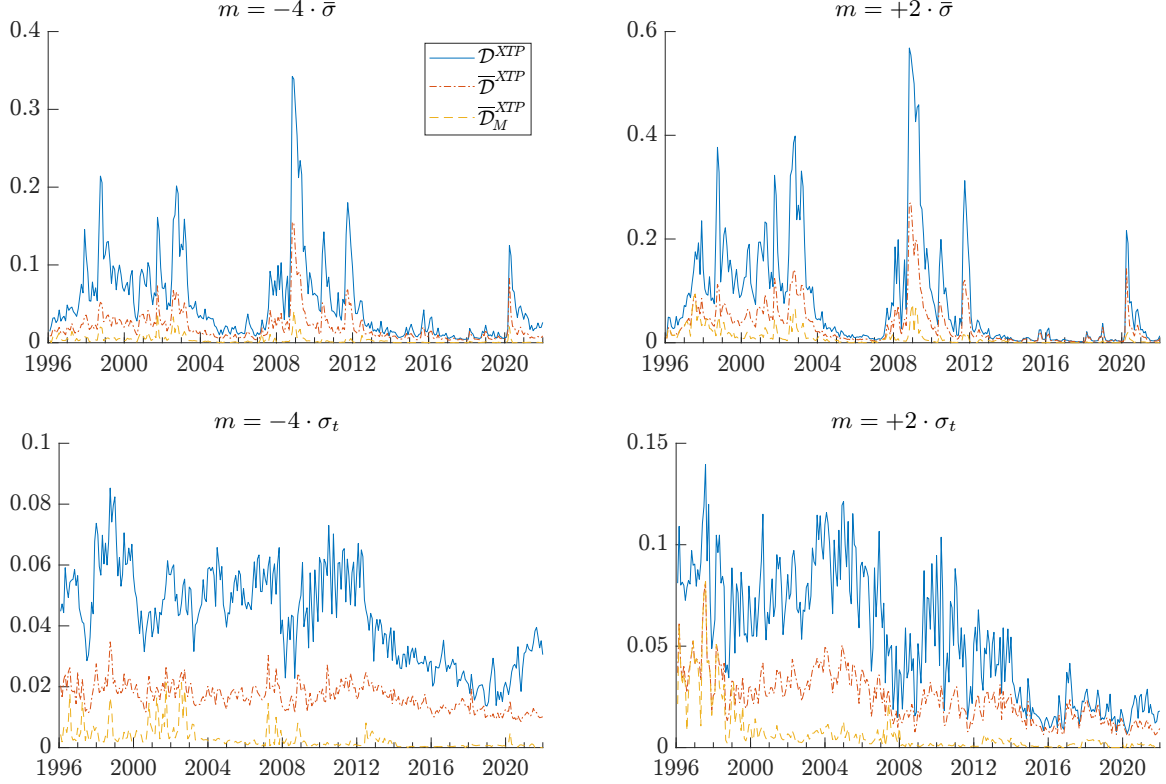


Figure 5.2: Time series of tail probability uncertainty ($\tau = 1$ -month)

This figure shows the time series of tail probability uncertainty $\mathcal{D}^{XTP}(e^m, \tau)$ at maturity $\tau = 1$ -month. The figure also shows the corresponding maturity-interpolated uncertainty $\bar{\mathcal{D}}^{XTP}(e^m, \tau)$ and mid uncertainty $\bar{\mathcal{D}}_M^{XTP}(e^m, \tau)$. The left column ($m < 0$) shows left-tail probabilities; the right column ($m > 0$) shows right-tail probabilities. The top row uses constant volatility scaling with $\bar{\sigma} = \sqrt{\tau} \bar{IV}^{ATM}(\tau)$; the bottom row uses time-varying volatility scaling with $\sigma_t = \sqrt{\tau} IV_t^{ATM}(\tau)$, where $IV_t^{ATM}(\tau)$ is the interpolated at-the-money mid implied volatility for maturity τ on date t and $\bar{IV}^{ATM}(\tau)$ is its sample average. The plotted uncertainties are monthly averages of daily uncertainties.

As the bounds themselves in Figure 4.6, also tail probability uncertainty for constant and time-varying m displays structural differences in their time series behavior. In the constant case, all measures of uncertainty exhibit a pronounced co-movement with market volatility. In contrast, in the time-varying case, there is no such clear cyclical pattern. Model-free and maturity-interpolated bounds are numerically large, even in the more recent part of the sample period. These bounds suggest that maturity incompleteness still plays an important role, but not as dominant as for implied volatilities.

So far, we have focused the analysis of tail probability uncertainty on a single maturity and a single moneyness level in each tail. Figure 5.3 provides insights into the commonalities of uncertainty at different moneyness levels and maturities. For this purpose, we restrict the attention to maturity-interpolated uncertainty when employing time-varying m .

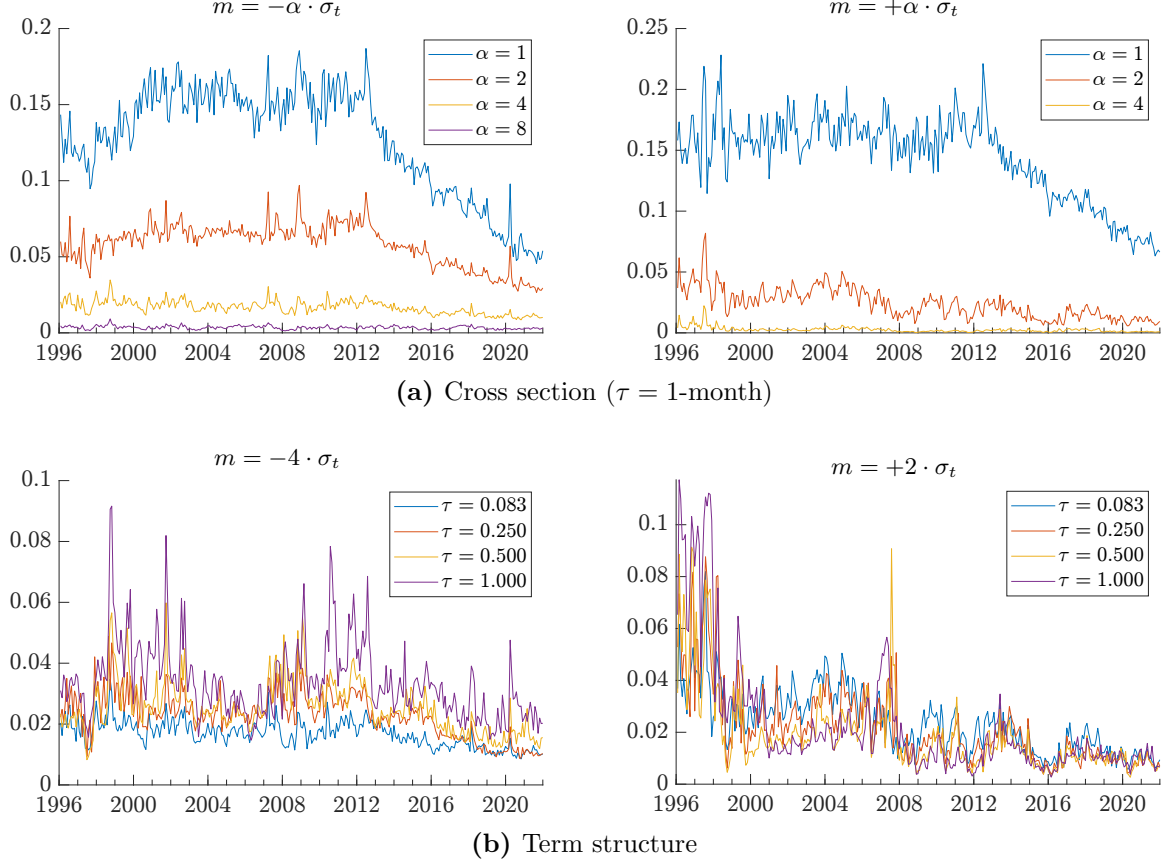


Figure 5.3: Cross section and term structure of tail probability uncertainty

This figure shows the time series of maturity-interpolated tail probability uncertainty $\bar{D}^{XTP}(e^m, \tau)$ for different combinations of m and τ . The left column ($m < 0$) shows left-tail probabilities; the right column ($m > 0$) shows right-tail probabilities. All plots use time-varying volatility scaling with $\sigma_t = \sqrt{\tau} IV_t^{ATM}(\tau)$, where $IV_t^{ATM}(\tau)$ is the interpolated at-the-money mid implied volatility for maturity τ on date t . The plotted uncertainties are monthly averages of daily uncertainties.

Panel (a) confirms the prior observation that tail probability uncertainty, in absolute terms, decreases for larger absolute moneyiness levels. For $|\alpha| = 1$, the uncertainty in either of the tails amounts to more than 15% over vast parts of the sample period. Throughout, the level of uncertainty decreases monotonously as the absolute moneyiness level increases. Notably, uncertainty across different moneyiness levels exhibits broadly similar time series patterns. However, the downward trend in uncertainty beginning around 2012 is considerably more pronounced for smaller absolute moneyiness levels than for larger ones.

Panel (b) additionally shows that tail probability uncertainty across different maturities moves largely in parallel at the given moneyiness levels.

5.3 Uncertainty across states of the economy

The option price and tail probability uncertainty in sections 5.1 and 5.2 measure the state of the option market in a broad sense. They mechanically capture the evolution of the option market as such, specifically the strike-maturity coverage of available option contracts and their embedded

transaction costs. Beyond that, they can also be expected to convey dynamic information on the fundamental disagreement of market participants about the underlying quantities.

Table 5.1 reports the correlation between maturity-interpolated tail probability uncertainty at 1-month maturity, evaluated at time-varying m , and various economic variables across several different categories. Overall, we find that this tail probability uncertainty is not particularly strongly related to any of the economic variables considered. The strongest (positive) relation exists with respect to the consumption-wealth ratio (cay) of Lettau and Ludvigson (2001), which rises in bad times.

Table 5.1: Tail probability uncertainty across states of the economy

This table shows the correlation between maturity-interpolated tail probability uncertainty $\overline{D}^{XTP}(e^m, \tau)$ at $\tau = 1$ -month and various economic variables. The column ρ_{lin} reports the Pearson linear correlation coefficient; the column ρ_{rank} reports the Spearman rank correlation coefficient. The employed uncertainties are monthly averages of daily uncertainties.

	$m = -4 \cdot \sigma_t$		$m = +2 \cdot \sigma_t$	
	ρ_{lin}	ρ_{rank}	ρ_{lin}	ρ_{rank}
General market conditions				
Dividend-price ratio	−0.16	−0.14	−0.26	−0.21
Consumption-wealth ratio (cay)	0.45	0.44	0.51	0.55
Term spread	0.27	0.19	0.25	0.17
Credit spread	−0.22	−0.19	−0.39	−0.32
Default spread	−0.00	−0.03	−0.13	−0.12
Market volatility				
Market vol (VIX)	0.15	0.11	−0.09	−0.02
Market vol of vol (VVIX)	−0.17	−0.11	−0.47	−0.34
Financial and macroeconomic uncertainty				
Financial uncertainty	−0.04	−0.03	−0.14	−0.09
Macroeconomic uncertainty	−0.21	−0.08	−0.26	−0.11
News-based market uncertainty	0.05	0.04	−0.03	−0.00
News-based policy uncertainty	−0.11	−0.09	−0.23	−0.15
Market maker constraints				
Intermediary capital risk	0.01	−0.01	0.21	0.14
Forecast dispersion				
Equity market disagreement	0.20	0.10	0.17	0.16
Analyst disagreement	−0.19	−0.12	−0.43	−0.28
GDP disagreement	−0.21	−0.06	−0.21	−0.09

6 Further implications for empirical asset pricing

We briefly discuss the implications of our results for general option-implied moments, which we further investigate in Bondarenko et al. (2024) and Dillschneider (2025).

It is well-known that a broad class of payoffs $f(R_t)$ can be spanned by the family of option

payoffs in R_t with strikes $k \in (0, \infty)$ and maturity $t \in \mathcal{T}$ (see originally Bick (1982) and later Carr and Madan (2001a,b)). Under weak regularity conditions, given some $\mathbb{Q}_t$, a twice continuously differentiable function f obeys the static replication formula

$$\mathbb{E}^{\mathbb{Q}_t}[f(R_t)] = f(1) + \int_0^1 f''(k) P^{\mathbb{Q}_t}(k, t) dk + \int_1^\infty f''(k) C^{\mathbb{Q}_t}(k, t) dk. \quad (6.1)$$

Empirical asset pricing applications regularly employ approximations of equation (6.1) with $P^{\mathbb{Q}_t}(k, t)$ and $C^{\mathbb{Q}_t}(k, t)$ replaced by observed option prices. In a hypothetical complete option market, this common procedure yields a unique "option-implied" moment $\mathbb{E}^{\mathbb{Q}_t}[f(R_t)]$ without explicitly recovering the associated probability measure $\mathbb{Q}_t$. Widely used examples in the literature include option-based measures of return variance (such as VIX-type indices) or higher-order return moments (e.g., Bakshi et al. (2003), Dew-Becker (2024), and Gormsen and Jensen (2025)). Such option-implied moments are commonly regarded as being measured with high precision (e.g., Jiang and Tian (2005)).

Taking option market incompleteness serious within the framework of this paper, each option-implied moment $\mathbb{E}^{\mathbb{Q}_t}[f(R_t)]$ generally becomes ambiguous. In order to quantify the degree of uncertainty inherent in $\mathbb{E}^{\mathbb{Q}_\tau}[f(R_\tau)]$ at maturity $\tau \in \mathcal{T}$ for a sufficiently well-behaved f , one may determine lower and upper moment bounds from the problems

$$\inf_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} \mathbb{E}^{\mathbb{Q}_\tau}[f(R_\tau)]. \quad (6.2)$$

The resulting lower bound (infimum) is denoted by $\mathcal{L}(f, \tau)$ and the upper bound (supremum) by $\mathcal{U}(f, \tau)$. Clearly, problems (3.4) and (3.21) are special cases.

Moment bounds arising from problems (6.2) are investigated in depth in Bondarenko et al. (2024) within a simplified single-period setting and in Dillschneider (2025) within a general multi-period setting. In both cases, option market incompleteness is found to have profound consequences for option-implied moments. Essentially, most moments employed in the existing literature are found to admit infinite bounds, thus failing even the most basic robustness requirement for empirical work.

The option price bounds established in section 3 are useful for quantifying general moment bounds and essential for understanding their behavior. On an intuitive level, this can be directly seen from the static replication formula (6.1) when the payoff function f is convex (i.e., $f''(k) \geq 0$). In that case, the lower and upper bounds satisfy

$$\mathcal{L}(f, \tau) \geq f(1) + \int_0^1 f''(k) \mathcal{L}^P(k, \tau) dk + \int_1^\infty f''(k) \mathcal{L}^C(k, \tau) dk, \quad (6.3)$$

$$\mathcal{U}(f, \tau) \leq f(1) + \int_0^1 f''(k) \mathcal{U}^P(k, \tau) dk + \int_1^\infty f''(k) \mathcal{U}^C(k, \tau) dk, \quad (6.4)$$

in terms of the option price bounds in Proposition 3.1. In most applications, the inequalities in

equations (6.3) and (6.4) will be rather tight.¹⁷ Hence, these relations suggest that the finiteness of moment bounds depends crucially on the relative behavior of payoff functions and option price bounds in the tails. This intuition effectively underlies our main insights in Bondarenko et al. (2024). In Dillschneider (2025), we formalize the construction of moment bounds via generalizations of the right-hand sides in equations (6.3) and (6.4), exploiting the option price bounds in Proposition 3.1, and show that the intuition also carries over more generally.

7 Conclusion

Absent strong (parametric) restrictions, a finite set of bid and ask option prices is generally compatible with infinitely many risk-neutral probability measures, leading to a range of feasible arbitrage-free European option prices at any given strike. This paper provides explicit model-free bounds for option prices that are consistent with a finite sample of bid and ask option prices, while otherwise imposing only elementary no-arbitrage conditions in a multi-period setting. Beyond the case of European option prices, we also derive explicit bounds for left-tail and right-tail probabilities of the underlying asset price, depending directly on the expressions for option price bounds. The widths of these bounds provide measures of uncertainty about the respective unobserved "true" quantities, for which we suggest a decomposition into contributions of incompleteness in the maturity, strike, and quoted price dimensions.

In an empirical analysis using S&P 500 index options, we quantify both option price bounds and tail probability bounds and investigate the resulting uncertainty. We document an important role for uncertainty due to maturity incompleteness, which regularly arises in the common research setting in which one measures option-based information at constant maturity, while observed maturities are time-varying. Moreover, we observe substantial uncertainty especially for left-tail probability levels, even when removing the prominent maturity component. This finding challenges the common belief that this kind of information can be measured from options at high precision. The actual uncertainty involved in tail probabilities interferes with many applications for which this information has been used in the existing literature. By uncovering these issues, we complement the analysis in Bondarenko et al. (2024), which focuses on issues with more general "option-implied" moments.

¹⁷In fact, equality holds in equation (6.4) when the integrals are understood in a certain limiting sense. Moreover, equality holds in equation (6.3) when considering continuous strike intervals with zero bid-ask spreads.

Appendix

A Economic interpretation of option price bounds

Consider the call problems (3.4) for an option specification (κ, τ, C) ; the respective put problems may be treated analogously. The associated dual problems can be written in the following form:

$$\begin{aligned}
 \sup / \inf_{\{\lambda_t\}} \quad & \sum_{t \in \mathbb{T}} \left(\lambda_t^1 + \lambda_t^R + \sum_{k \in \mathbb{K}} \alpha (\lambda_{k,t}^C)^- B^C(k, t) + \alpha (\lambda_{k,t}^C)^+ A^C(k, t) \right) \\
 \text{s.t.} \quad & \sum_{t \in \mathbb{T}} \left(\lambda_t^1 + \lambda_t^R R_t + \sum_{k \in \mathbb{K}} \alpha \lambda_{k,t}^C (R_t - k)^+ \right) \\
 & + \sum_{t < t' \in \mathcal{T}} \left(\sum_{k \in \Omega} \alpha \lambda_{k,t,t'}^M ((R_t - k)^+ - (R_{t'} - k)^+) \right) \\
 & \leq / \geq (R_\tau - \kappa)^+, \quad R_\tau \in \Omega^\mathcal{T}, \\
 & \lambda_{k,t,t'}^M \geq 0, \quad \alpha = -1 / +1.
 \end{aligned} \tag{A.1}$$

Here, the inequality $f(R_\tau) = \sum_{t \in \mathcal{T}} f_t(R_t) \leq / \geq g(R_\tau)$ for $R_\tau \in \Omega^\mathcal{T}$ should be understood dimensionwise, i.e., equivalent to $f_\tau(R_\tau) \leq / \geq g(R_\tau)$ and $f_t(R_t) \leq / \geq 0$ for any $t \neq \tau$. To be clear, the employed notation means that the supremum problem uses a $\leq$ constraint with $\alpha = -1$, whereas the infimum problem uses a $\geq$ constraint with $\alpha = +1$. Provided that strong duality holds, the solutions of the dual problems (A.1) coincide with those of the primal problems (3.4). Concretely, denoting the solutions of the dual problems by $\tilde{\mathcal{U}}^C(\kappa, \tau)$ and $\tilde{\mathcal{L}}^C(\kappa, \tau)$, respectively, we find that $\mathcal{L}^C(\kappa, \tau) = \tilde{\mathcal{U}}^C(\kappa, \tau)$ and $\mathcal{U}^C(\kappa, \tau) = \tilde{\mathcal{L}}^C(\kappa, \tau)$.

Economically, the dual problems (A.1) construct the most expensive/cheapest portfolio consisting of a risk-free asset, the risky underlying asset, and call options with $(k, t) \in \mathbb{K} \times \mathbb{T}$ that sub-/super-replicate the option payoff $(R_\tau - \kappa)^+$. More precisely, the upper bound arises from super-replication of $(R_\tau - \kappa)^+$ from a trader's perspective (buying options at the ask and selling at the bid), while the lower bound is determined by sub-replication of $(R_\tau - \kappa)^+$ from a market maker's perspective (buying options at the bid and selling at the ask). By Proposition 3.1, the optimal super- or sub-replicating portfolios can be determined from the following basic call option strategies with at most two options, which differ from the option spread strategies considered in Cousot (2007):

- *Upper Long Call*: trader portfolio consisting of a unit long position in a single call at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (in the underlying when $k' = 0$) with $k' \leq \kappa$, $t' \geq \tau$,

$$LC^{\mathcal{U}}(\kappa, \tau; k', t') = A^C(k', t').$$

- *Upper Long-Long Calls*: trader portfolio consisting of a long position of $\frac{k'' - \kappa}{k'' - k'}$ in a call at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (in the underlying when $k' = 0$) and a long position of $\frac{\kappa - k'}{k'' - k'}$ in

another call at $(k'', t'') \in \mathbb{K} \times \mathbb{T}$ with $k' \leq \kappa < k''$, $t', t'' \geq \tau$,

$$LLC^{\mathcal{U}}(\kappa, \tau; k', t'; k'', t'') = \frac{k'' - \kappa}{k'' - k'} A^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'').$$

- *Lower Long Call*: market maker portfolio consisting of a unit long position in a single call at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (in the underlying when $k' = 0$) with $\kappa \leq k'$, $t' \leq \tau$,

$$LC^{\mathcal{L}}(\kappa, \tau; k', t') = B^C(k', t').$$

- *Lower Long-Short Calls*: market maker portfolio consisting of a long position of $\frac{k'' - \kappa}{k'' - k'}$ in a call at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (in the underlying when $k' = 0$) and a short position of $\frac{\kappa - k'}{k'' - k'}$ in another call at $(k'', t'') \in \mathbb{K} \times \mathbb{T}$ with $\kappa \leq k' < k''$, $t' \leq t'', \tau$,

$$LSC^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') = \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'').$$

- *Lower Short-Long Calls*: market maker portfolio consisting of a short position of $\frac{k'' - \kappa}{k'' - k'}$ in a call at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (in the underlying when $k' = 0$) and a long position of $\frac{\kappa - k'}{k'' - k'}$ in another call at $(k'', t'') \in \mathbb{K} \times \mathbb{T}$ with $k' < k'' \leq \kappa$, $t'' \leq t', \tau$,

$$SLC^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') = \frac{k'' - \kappa}{k'' - k'} A^C(k', t') + \frac{\kappa - k'}{k'' - k'} B^C(k'', t'').$$

Equivalently, using put-call parity, the optimal super- or sub-replicating portfolios can be determined from the following basic put option strategies with at most two options:

- *Upper Long Put*: trader portfolio consisting of a long position of $\kappa - k'$ in the risk-free asset and a unit long position in a put at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (no position when $k' = 0$) with $k' \leq \kappa$, $t' \geq \tau$,

$$LP^{\mathcal{U}}(\kappa, \tau; k', t') = A^P(k', t') + \kappa - k'.$$

- *Upper Long-Long Puts*: trader portfolio consisting of a long position of $\frac{k'' - \kappa}{k'' - k'}$ in a put at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (no position when $k' = 0$) and a long position of $\frac{\kappa - k'}{k'' - k'}$ in another put at $(k'', t'') \in \mathbb{K} \times \mathbb{T}$ with $k' \leq \kappa < k''$, $t', t'' \geq \tau$,

$$LLP^{\mathcal{U}}(\kappa, \tau; k', t'; k'', t'') = \frac{k'' - \kappa}{k'' - k'} A^P(k', t') + \frac{\kappa - k'}{k'' - k'} A^P(k'', t'').$$

- *Lower Long Put*: market maker portfolio consisting of a short position of $\kappa - k'$ in the risk-free asset and a unit long position in a put at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (no position when $k' = 0$) with $\kappa \leq k'$, $t' \leq \tau$,

$$LP^{\mathcal{L}}(\kappa, \tau; k', t') = B^P(k', t') + \kappa - k'.$$

- *Lower Long-Short Puts*: market maker portfolio consisting of a long position of $\frac{k'' - \kappa}{k'' - k'}$ in a put at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (no position when $k' = 0$) and a short position of $\frac{\kappa - k'}{k'' - k'}$ in another put at $(k'', t'') \in \mathbb{K} \times \mathbb{T}$ with $\kappa \leq k' < k''$, $t' \leq t'', \tau$,

$$LSP^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') = \frac{k'' - \kappa}{k'' - k'} B^P(k', t') + \frac{\kappa - k'}{k'' - k'} A^P(k'', t'').$$

- *Lower Short-Long Puts*: market maker portfolio consisting of a short position of $\frac{k'' - \kappa}{k'' - k'}$ in a put at $(k', t') \in (\mathbb{K} \cup \{0\}) \times \mathbb{T}$ (no position when $k' = 0$) and a long position of $\frac{\kappa - k'}{k'' - k'}$ in another put at $(k'', t'') \in \mathbb{K} \times \mathbb{T}$ with $k' < k'' \leq \kappa$, $t'' \leq t', \tau$,

$$SLP^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') = \frac{k'' - \kappa}{k'' - k'} A^P(k', t') + \frac{\kappa - k'}{k'' - k'} B^P(k'', t'').$$

Figure A.1 visualizes the sub- and super-replicating properties of the basic call option strategies with respect to the payoff $(R_\tau - \kappa)^+$ for the case where $t' = t'' = \tau$.

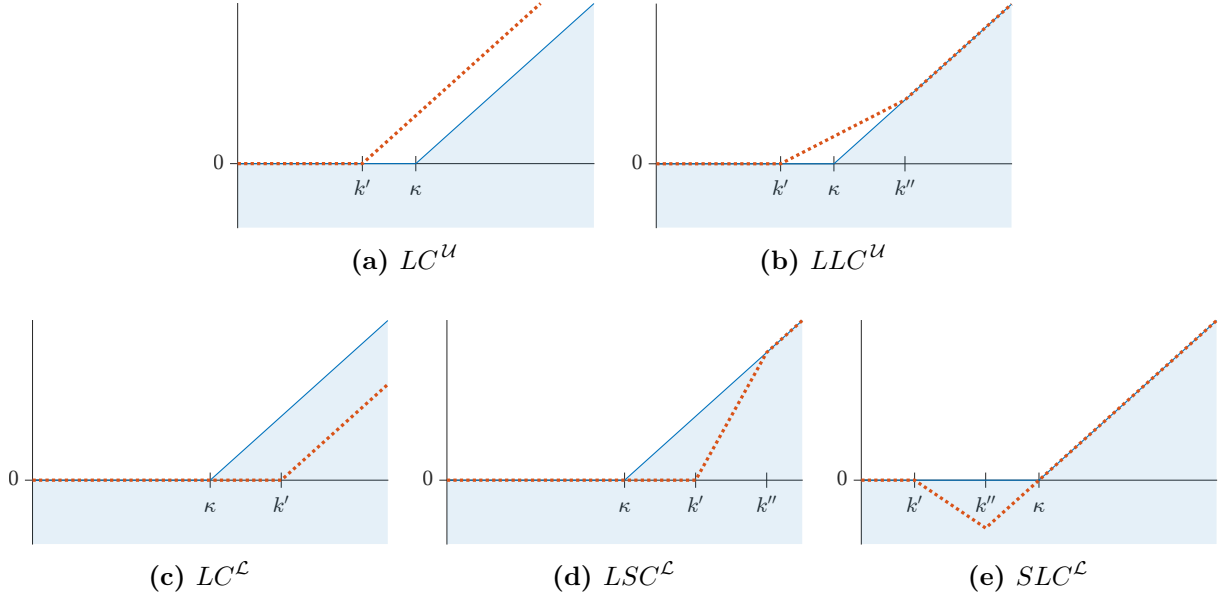


Figure A.1: Sub- and super-replication payoffs

Proposition 3.1 can be reformulated using the basic option strategies. Notably, the trader portfolios determine the upper bounds, while the market maker portfolios determine the lower bounds.

Corollary A.1. *Under the conditions of Proposition 3.1, the following holds in problems (3.4) for $k', k'' \in \mathbb{K} \cup \{0\}$ and $t', t'' \in \mathbb{T}$:*

(i) The bounds for European calls with $\kappa \geq 0$, $\tau > 0$ are given by

$$\begin{aligned} \mathcal{U}^C(\kappa, \tau) &= \inf\{LC^{\mathcal{U}}(\kappa, \tau; k', t') : k' \leq \kappa, t' \geq \tau\} \\ &\quad \wedge \inf\{LLC^{\mathcal{U}}(\kappa, \tau; k', t'; k'', t'') : k' \leq \kappa < k'', t', t'' \geq \tau\}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \mathcal{L}^C(\kappa, \tau) &= (1 - \kappa)^+ \vee \sup\{LC^{\mathcal{L}}(\kappa, \tau; k', t') : \kappa \leq k', t' \leq \tau\} \\ &\quad \vee \sup\{LSC^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') : \kappa \leq k' < k'', t' \leq t'', \tau\} \\ &\quad \vee \sup\{SLC^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') : k' < k'' \leq \kappa, t'' \leq t', \tau\}. \end{aligned} \quad (\text{A.3})$$

(ii) The bounds for European puts with $\kappa \geq 0$, $\tau > 0$ are given by

$$\begin{aligned} \mathcal{U}^P(\kappa, \tau) &= \inf\{LP^{\mathcal{U}}(\kappa, \tau; k', t') : k' \leq \kappa, t' \geq \tau\} \\ &\quad \wedge \inf\{LLP^{\mathcal{U}}(\kappa, \tau; k', t'; k'', t'') : k' \leq \kappa < k'', t', t'' \geq \tau\}, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \mathcal{L}^P(\kappa, \tau) &= (\kappa - 1)^+ \vee \sup\{LP^{\mathcal{L}}(\kappa, \tau; k', t') : \kappa \leq k', t' \leq \tau\} \\ &\quad \vee \sup\{LSP^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') : \kappa \leq k' < k'', t' \leq t'', \tau\} \\ &\quad \vee \sup\{SLP^{\mathcal{L}}(\kappa, \tau; k', t'; k'', t'') : k' < k'' \leq \kappa, t'' \leq t', \tau\}. \end{aligned} \quad (\text{A.5})$$

Online Appendix

OA Auxiliary results

This appendix reviews no-arbitrage conditions for option price surfaces as well as their construction through lower convex envelopes, which are essential for the derivation of explicit option price bounds.

OA.1 Risk-neutral probability measures

By the Fundamental Theorem of Asset Pricing, $(R_t : t \in \mathcal{T})$ is a $\mathbb{Q}$ -martingale under any admissible risk-neutral probability measure $\mathbb{Q}$. Among others, it thus needs to hold that $E^{\mathbb{Q}_t}[R_t] = R_0 = 1$ for all $t \in \mathcal{T}$. Denoting by $\mathcal{M}(\Omega)$ the set of positive Borel measures on the state space Ω , we thus define the set of risk-neutral marginal probability measures

$$\mathcal{M}_1(\Omega) := \left\{ \mu \in \mathcal{M}(\Omega) : E^\mu[1] = 1, E^\mu[R] = 1 \right\}. \quad (\text{OA.1})$$

Every $\mathbb{Q}_t \in \mathcal{M}_1(\Omega)$ is a probability measure by integrating to unity and reflects the martingale property by having unit expectation $E^{\mathbb{Q}_t}[R_t]$.

To assure time consistency, we invoke the notion of convex order of $\mathbb{Q}_{\mathcal{T}} = (\mathbb{Q}_t : t \in \mathcal{T})$, i.e., the property that expectations with respect to $\mathbb{Q}_t$ are increasing in t for any convex function. The convex order of $\mathbb{Q}_{\mathcal{T}}$ is equivalent to the requirement that, for all $k \in \Omega$ and $t, t' \in \mathcal{T}$ with $t < t'$, we have (e.g., by Corollary 2.61 in Föllmer and Schied (2016))

$$C^{\mathbb{Q}_{\mathcal{T}}}(k, t) := E^{\mathbb{Q}_t}[(R_t - k)^+] \leq E^{\mathbb{Q}_{t'}}[(R_{t'} - k)^+] =: C^{\mathbb{Q}_{\mathcal{T}}}(k, t'). \quad (\text{OA.2})$$

Hence, intuitively, it is necessary and sufficient for $\mathbb{Q}_{\mathcal{T}}$ to produce call option prices $C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ that are increasing with time-to-maturity t for any positive strike k .

Making use of theory developed in Kellerer (1972) and Strassen (1965) and recently extended in Hirsch et al. (2011) and Hirsch and Roynette (2012), the convex order of $(\mathbb{Q}_t : t \in \mathcal{T})$ can be shown to be equivalent to the existence of a mimicking martingale process. Specifically, provided the convex order, there exists some martingale $(M_t : t \in \mathcal{T})$ associated to $(R_t : t \in \mathcal{T})$ such that M_t agrees in law with R_t at each $t \in \mathcal{T}$. We summarize this for our purposes auxiliary result in the following lemma, which corresponds to Theorem 1.1 in Hirsch et al. (2011).¹⁸

Lemma OA.1. *The collection of probability measures $(\mathbb{Q}_t : t \in \mathcal{T}) \subset \mathcal{M}_1(\Omega)$ being in convex order is equivalent to the existence of a martingale $(M_t : t \in \mathcal{T})$ that agrees in law with $(R_t : t \in \mathcal{T})$ univariately at each $t \in \mathcal{T}$.*

Lemma OA.1 leads us to define the basic set of risk-neutral probability measures

$$\mathcal{Q}(\mathcal{T}) := \left\{ \mathbb{Q}_{\mathcal{T}} \subset \mathcal{M}_1(\Omega) : C^{\mathbb{Q}_{\mathcal{T}}}(k, t) \leq C^{\mathbb{Q}_{\mathcal{T}}}(k, t') \ \forall k \in \Omega, t < t' \in \mathcal{T} \right\}, \quad (\text{OA.3})$$

which is equivalent to the definition stated in section 3.1.

¹⁸On a discrete state space, a related result in the form of the Sherman-Stein-Blackwell Theorem is used in Davis and Hobson (2007).

OA.2 Arbitrage-free European option prices

By the term call option price surface, we mean a mapping $(k, t) \mapsto C(k, t)$ that assigns an option price to each combination of strike and time-to-maturity. Following Davis and Hobson (2007), we refer to such an option price surface as (statically) arbitrage-free on $\Omega \times \mathcal{T}$ in the model-independent sense if there exists a $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$ with $C^{\mathbb{Q}_{\mathcal{T}}}(k, t) = C(k, t)$ for all $(k, t) \in \Omega \times \mathcal{T}$. The association between an arbitrage-free option price surface C and the respective $\mathbb{Q}_{\mathcal{T}}$ is unique. Given an arbitrage-free $C(\cdot, t)$, we may directly reconstruct the option-implied probability measure $\mathbb{Q}_t$ for $t \in \mathcal{T}$ via

$$\mathbb{Q}_t(R_t \leq k) = \mathbb{E}^{\mathbb{Q}_t}[\mathbb{1}_{[0, k]}(R_t)] = 1 + \partial_{k+} C(k, t). \quad (\text{OA.4})$$

In particular, the point mass located at k is $\mathbb{Q}_t(R_t = k) = \partial_{k+} C(k, t) - \partial_{k-} C(k, t)$, where we denote by ∂_{k+} the right-hand derivative operator with respect to the strike price and likewise by ∂_{k-} the corresponding left-hand derivative operator. Equation (OA.4) thus determines the cumulative distribution function, which uniquely characterizes $\mathbb{Q}_t$ and yields $\mathbb{Q}_{\mathcal{T}} = (\mathbb{Q}_t : t \in \mathcal{T}) \in \mathcal{Q}(\mathcal{T})$.

Necessary and sufficient conditions for arbitrage-free option price curves and surfaces have been established in the existing literature. Using results in Ait-Sahalia and Duarte (2003), Carr and Madan (2005), Cousot (2007), Davis and Hobson (2007), and Roper (2010), among others, which are translated in Fengler and Hin (2015) to the forward return setting employed here, the following lemma summarizes the main conditions for future reference.

Lemma OA.2. *An option price surface C is (statically) arbitrage-free with $C(k, t) = C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ for all $(k, t) \in \Omega \times \mathcal{T}$ and some $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$ if and only if it satisfies the following conditions for any $k, k' \in \Omega$ and $t, t' \in \mathcal{T}$:*

- (i) *bounded:* $(1 - k)^+ \leq C(k, t) \leq 1$ with $C(k, 0) = (1 - k)^+$, $C(0, t) = 1$, $\lim_{k \rightarrow \infty} C(k, t) = 0$;
- (ii) *decreasing in k :* $-1 \leq \partial_{k+} C(k, t) \leq 0$;
- (iii) *convex in k :* $\partial_{k+} C(k, t) \leq \partial_{k+} C(k', t)$ whenever $k < k'$;
- (iv) *increasing in t :* $C(k, t) \leq C(k, t')$ whenever $t < t'$.

In Lemma OA.2, condition (iii) entails that $C(\cdot, t)$ is continuous for all t . To formulate the decreasing and convex behavior in conditions (ii) and (iii), we may equivalently use right-hand and left-hand derivative operators ∂_{k+} and ∂_{k-} , respectively, whose existence is assured provided convexity of $C(\cdot, t)$ (e.g., Theorem 23.1 in Rockafellar (1970)). Specifically, using $\partial_{k-} C(0, t) = -1$, condition (ii) may also be expressed as $-1 \leq \partial_{k-} C(k, t) \leq 0$, while condition (iii) can be written as $\partial_{k-} C(k, t) \leq \partial_{k-} C(k', t)$ for all $k < k'$ and moreover yields that $\partial_{k-} C(k, t) \leq \partial_{k+} C(k, t)$ for each k . Accounting for the limiting behavior as $k \rightarrow \infty$ required by condition (i), Lemma OA.2 thereby implies that $\partial_{k+} C(k, t) < 0$ whenever $C(k, t) > 0$. Condition (iv) further adds a time consistency requirement through the convex order condition in equation (OA.2).

Lemma OA.2 essentially characterizes an arbitrage-free option price surface through a sufficiently regular set of tuples $\{(k, t, C(k, t)) : (k, t) \in \Omega \times \mathcal{T}\}$. Dropping this regularity requirement, we may attempt to construct an arbitrage-free option price surface from some data $\mathbb{C} := \{(k, t, c(k, t)) : (k, t) \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}\}$ with arbitrary $c(k, t) \geq (1 - k)^+$, where $\tilde{\mathbb{K}} \subset \Omega$ is a set of strikes and $\tilde{\mathbb{T}} \subset \mathcal{T}$ is a set of maturities. We

define on $\Omega \times \mathcal{T}$ the lower convex envelope associated to $\mathbb{C}$ for $k', k'' \in \tilde{\mathbb{K}}$ and $t', t'' \in \tilde{\mathbb{T}}$ as

$$\begin{aligned} \mathcal{E}(\kappa, \tau; \mathbb{C}) &:= \{(1 - \kappa)^+ : \tau = 0\} \wedge \inf\{c(k', t') : k' \leq \kappa, t' \geq \tau\} \\ &\wedge \inf\left\{\frac{k' - \kappa}{k'} + \frac{\kappa}{k'} c(k', t') : 0 \leq \kappa < k', t' \geq \tau\right\} \\ &\wedge \inf\left\{\frac{k'' - \kappa}{k'' - k'} c(k', t') + \frac{\kappa - k'}{k'' - k'} c(k'', t'') : k' \leq \kappa < k'', t', t'' \geq \tau\right\}. \end{aligned} \quad (\text{OA.5})$$

We in particular observe that $\lim_{k \rightarrow \infty} \mathcal{E}(k, t; \mathbb{C}) = I(t) := \inf\{c(k', t') : (k', t') \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}, t' \geq t\}$. Unless $I(t) = 0$ holds for all $t \in \mathcal{T}$, $\mathcal{E}(k, t; \mathbb{C})$ fails to exhibit the necessary limiting behavior as $k \rightarrow \infty$ prescribed by Lemma OA.2. Under this assumption, $\mathcal{E}(\cdot, \cdot; \mathbb{C})$ satisfies all conditions in Lemma OA.2 and can thus be associated to a probability measure $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$ such that $\mathcal{E}(k, t; \mathbb{C}) = C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ for all $(k, t) \in \Omega \times \mathcal{T}$. Hence, $\mathcal{E}(\cdot, \cdot; \mathbb{C})$ constitutes an arbitrage-free option price surface. In fact, $\mathcal{E}(\cdot, \cdot; \mathbb{C})$ is the (pointwise) largest such surface satisfying $\mathcal{E}(k, t; \mathbb{C}) \leq c(k, t)$ for each $(k, t) \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}$. This yields the following lemma that will be essential for the explicit construction of arbitrage-free option price surfaces.

Lemma OA.3. *Let $\mathbb{C} := \{(k, t, c(k, t)) : (k, t) \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}\}$ such that $c(k, t) \geq (1 - k)^+$ for all $(k, t) \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}$. Moreover, let $I(t) := \inf\{c(k', t') : (k', t') \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}, t' \geq t\} = 0$ for all $t \in \mathcal{T}$. Then $\mathcal{E}(\cdot, \cdot; \mathbb{C})$ defines an arbitrage-free option price surface $C^{\mathbb{Q}_{\mathcal{T}}}$ on $\Omega \times \mathcal{T}$ for $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$.*

Proof. To prove the lemma, we show that $\mathcal{E}(\cdot, \cdot; \mathbb{C})$ satisfies the conditions in Lemma OA.2.

Regarding condition (i), note that $c(k, t) \geq (1 - k)^+$ for all $(k, t) \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}$. Take any $(\kappa, \tau) \in \Omega \times \mathcal{T}$. By convexity of $k \mapsto (1 - k)^+$, we have for any $k' \leq \kappa < k''$ and $t', t'' \geq \tau$ the relation

$$\frac{k'' - \kappa}{k'' - k'} c(k', t') + \frac{\kappa - k'}{k'' - k'} c(k'', t'') \geq \frac{k'' - \kappa}{k'' - k'} (1 - k')^+ + \frac{\kappa - k'}{k'' - k'} (1 - k'')^+ \geq (1 - \kappa)^+.$$

Hence, the infimum in equation (OA.5) is $\geq (1 - \kappa)^+$, so that $\mathcal{E}(\kappa, \tau; \mathbb{C}) \geq (1 - \kappa)^+$ holds. Moreover, by construction in equation (OA.5), $\mathcal{E}(\kappa, 0; \mathbb{C}) = (1 - \kappa)^+$ and $\mathcal{E}(0, \tau; \mathbb{C}) = 1$. The relation

$$\mathcal{E}(\kappa, \tau; \mathbb{C}) \leq \tilde{I}(\kappa, \tau) := \inf\{c(k', t') : (k', t') \in \tilde{\mathbb{K}} \times \tilde{\mathbb{T}}, t' \geq \tau, k' \leq \kappa\}$$

further implies that $\lim_{\kappa \rightarrow \infty} \mathcal{E}(\kappa, \tau; \mathbb{C}) \leq \lim_{\kappa \rightarrow \infty} \tilde{I}(\kappa, \tau) = I(\tau) = 0$. Therefore, the boundedness requirements in condition (i) are satisfied.

To establish condition (ii), fix any $\tau \in \mathcal{T}$ and suppose that $\mathcal{E}(\cdot, \tau; \mathbb{C})$ is convex as in condition (iii), which will be verified below. Due to convexity and since $\mathcal{E}(\kappa, \tau; \mathbb{C}) \geq 1 - \kappa$ as well as $\mathcal{E}(0, \tau; \mathbb{C}) = 1$, it holds that

$$\partial_{k+} \mathcal{E}(\kappa, \tau; \mathbb{C}) \geq \frac{\mathcal{E}(\kappa, \tau; \mathbb{C}) - \mathcal{E}(0, \tau; \mathbb{C})}{\kappa} = \frac{\mathcal{E}(\kappa, \tau; \mathbb{C}) - 1}{\kappa} \geq -1.$$

Similarly, it further follows by convexity and $\mathcal{E}(\kappa, \tau; \mathbb{C}) \geq 0$ as well as $\lim_{u \rightarrow \infty} \mathcal{E}(u, \tau; \mathbb{C}) = 0$ that

$$\partial_{k+} \mathcal{E}(\kappa, \tau; \mathbb{C}) \leq \lim_{k \rightarrow \infty} \frac{\mathcal{E}(k, \tau; \mathbb{C}) - \mathcal{E}(\kappa, \tau; \mathbb{C})}{k - \kappa} = \lim_{k \rightarrow \infty} -\frac{\mathcal{E}(k, \tau; \mathbb{C})}{k - \kappa} = 0.$$

Hence, provided convexity, $\mathcal{E}(\cdot, \tau; \mathbb{C})$ is decreasing as required by condition (ii).

To verify the convexity condition (iii), fix again any $\tau \in \mathcal{T}$. As convexity is obvious when $\tau = 0$, the

remaining argument only treats the non-trivial case $\tau > 0$. We introduce the linear program

$$\begin{aligned} \min_w \quad & \sum_{k,t} w_{k,t} c(k,t) \\ \text{s.t.} \quad & w_{k,t} \geq 0, \quad \sum_{k,t} w_{k,t} = 1, \quad \sum_{k,t} w_{k,t} k = \kappa \end{aligned} \tag{OA.6}$$

with summation over $(k,t) \in \tilde{\mathbb{K}}' \times \tilde{\mathbb{T}}'$, where $\tilde{\mathbb{K}}' \subset \tilde{\mathbb{K}}$ and $\tilde{\mathbb{T}}' \subset \tilde{\mathbb{T}}$ are taken to be finite sets. Problem (OA.6) is feasible for any $0 \leq \kappa < \bar{K} := \sup \tilde{\mathbb{K}}$, where $\bar{K}$ may be infinite. Provided feasibility, standard linear programming theory guarantees the existence of an optimal basic feasible solution such that $w_{k,t}^* = 0$ for all (k,t) except for possibly (k^*, t^*) and (k^{**}, t^{**}) , where we may take $k^* \leq \kappa < k^{**}$. For any other feasible solution, we observe that by construction of $\mathcal{E}(\kappa, \tau; \mathbb{C})$ in equation (OA.5) when $\tau \leq \min \tilde{\mathbb{T}}'$,

$$\sum_{k,t} w_{k,t} c(k,t) \geq \frac{k^{**} - \kappa}{k^{**} - k^*} c(k^*, t^*) + \frac{\kappa - k^*}{k^{**} - k^*} c(k^{**}, t^{**}) \geq \mathcal{E}(\kappa, \tau; \mathbb{C}). \tag{OA.7}$$

Since $\mathcal{E}(k, \tau; \mathbb{C}) = (1 - k)^+$ outside of $(0, \bar{K})$, $\mathcal{E}(k, \tau; \mathbb{C}) \geq (1 - k)^+$ on $(0, \bar{K})$ and continuous at 0 and $\bar{K}$, it suffices to show that $\mathcal{E}(\cdot, \tau; \mathbb{C})$ is convex on $(0, \bar{K})$. Thus, take $\kappa, k', k'' \in (0, \bar{K})$ with $k' \leq \kappa < k''$. Due to the construction in equation (OA.5), for any arbitrary $\varepsilon > 0$, there exist $\tilde{k}' \leq k' < \tilde{k}''$ and $\hat{k}' \leq k'' < \hat{k}''$ as well as $\tilde{t}', \tilde{t}'', \hat{t}', \hat{t}'' \geq \tau$ such that

$$\mathcal{E}(k', \tau; \mathbb{C}) + \varepsilon \geq \frac{\tilde{k}'' - k'}{\tilde{k}'' - \tilde{k}'} c(\tilde{k}', \tilde{t}') + \frac{k' - \tilde{k}'}{\tilde{k}'' - \tilde{k}'} c(\tilde{k}'', \tilde{t}'), \tag{OA.8}$$

$$\mathcal{E}(k'', \tau; \mathbb{C}) + \varepsilon \geq \frac{\hat{k}'' - \kappa}{\hat{k}'' - \hat{k}'} c(\hat{k}', \hat{t}') + \frac{\kappa - \hat{k}'}{\hat{k}'' - \hat{k}'} c(\hat{k}'', \hat{t}''). \tag{OA.9}$$

Any convex combination of the right-hand sides of equations (OA.8) and (OA.9) yields some feasible $\sum_{k,t} w_{k,t} c(k,t)$ over $\tilde{\mathbb{K}}' := \{\tilde{k}', \tilde{k}'', \hat{k}', \hat{k}''\}$ and $\tilde{\mathbb{T}}' := \{\tilde{t}', \tilde{t}'', \hat{t}', \hat{t}''\}$. Therefore, combining these relations with equation (OA.7) yields

$$\frac{k'' - \kappa}{k'' - k'} \mathcal{E}(k', \tau; \mathbb{C}) + \frac{\kappa - k'}{k'' - k'} \mathcal{E}(k'', \tau; \mathbb{C}) + \varepsilon \geq \mathcal{E}(\kappa, \tau; \mathbb{C}). \tag{OA.10}$$

Letting $\varepsilon \rightarrow 0$, we obtain the convexity of $\mathcal{E}(\cdot, \tau; \mathbb{C})$. In conclusion, we arrive at the convexity property in condition (iii).

Finally, condition (iv) is a direct consequence of the construction in equation (OA.5). Concretely, for any κ , $\mathcal{E}(\kappa, \tau; \mathbb{C}) \leq \mathcal{E}(\kappa, \tau'; \mathbb{C})$ whenever $\tau < \tau'$ since the sets of admissible tuples $(k', t'), (k'', t'')$ within the infimum in equation (OA.5) are decreasing. $\square$

OB Proofs of main results

OB.1 Proof of Proposition 3.1

For proving the proposition, it suffices to derive the bounds for calls in equations (3.6) and (3.7). It is straightforward to verify that the corresponding bounds for puts in equations (3.8) and (3.9) obtain using the put-call parity relations (3.5).

Call price, upper bound: Denote the right-hand side in equation (3.6) by $U^C(\kappa, \tau)$. The proof proceeds in two steps. First, we show that $U^C(\kappa, \tau)$ is some upper bound for attainable call prices, satisfying $C^{\mathbb{Q}_T}(\kappa, \tau) \leq U^C(\kappa, \tau)$ for all $\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})$. Second, we verify that it is indeed the optimal upper bound (supremum) $\mathcal{U}^C(\kappa, \tau)$ in the call problem (3.4) by explicitly constructing a sequence $\{\mathbb{Q}_T^u\} \subset \mathcal{O}(\mathcal{T})$, parameterized by u , that approaches the upper bound such that $C^{\mathbb{Q}_T^u}(\kappa, \tau) \rightarrow U^C(\kappa, \tau)$ as $u \rightarrow \infty$. Since the case $\tau = 0$ is obvious, we suppose for the following arguments that $\tau > 0$.

To establish that $U^C(\kappa, \tau)$ is an upper bound, note that the convexity of $k \mapsto (1 - k)^+$ implies for any $k', k'' \in \mathbb{K} \cup \{0, \infty\}$ with $k' \leq \kappa < k''$ the "super-replicating" relation

$$(1 - \kappa)^+ \leq \frac{k'' - \kappa}{k'' - k'}(1 - k')^+ + \frac{\kappa - k'}{k'' - k'}(1 - k'')^+. \quad (\text{OB.1})$$

Here, any infinite k'' is understood in the limiting sense, thus yielding $(1 - \kappa)^+ \leq (1 - k')^+$ for any $k' \leq \kappa$. Each feasible $\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})$ specifically satisfies the constraints $C^{\mathbb{Q}_T}(k, \tau) \leq C^{\mathbb{Q}_T}(k, t) \leq A^C(k, t)$ for $k \in \mathbb{K} \cup \{0, \infty\}$ and any $t \in \mathbb{T}$ with $t \geq \tau$, where we set $A^C(0, t) = 1$ and $A^C(\infty, t) = 0$ for convenience. It therefore follows from equation (OB.1) that for every $t', t'' \geq \tau$, the inequalities

$$\begin{aligned} C^{\mathbb{Q}_T}(\kappa, \tau) &\leq \frac{k'' - \kappa}{k'' - k'} C^{\mathbb{Q}_T}(k', \tau) + \frac{\kappa - k'}{k'' - k'} C^{\mathbb{Q}_T}(k'', \tau) \\ &\leq \frac{k'' - \kappa}{k'' - k'} A^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') \end{aligned} \quad (\text{OB.2})$$

hold whenever $\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})$. Note that $U^C(\kappa, \tau)$ corresponds to the infimum of the right-hand side in equation (OB.2). It must therefore hold that $C^{\mathbb{Q}_T}(\kappa, \tau) \leq U^C(\kappa, \tau)$ for every $\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})$.

Having thereby shown that $U^C(\kappa, \tau)$ is an upper bound, we now explicitly construct an approaching sequence of probability measures in $\mathcal{O}(\mathcal{T})$. By assumption, there exists $\mathbb{Q}_T^* \in \mathcal{O}(\mathcal{T})$. For $(k, t) \in \Omega \times \mathcal{T}$ and some large $u > \kappa$, define for $u' \geq u$ with $k', k'' \in \mathbb{K}$ and $t', t'' \in \mathbb{T}$

$$\begin{aligned} U^C(k, t; u) &:= \{(1 - k)^+ : t = 0\} \wedge \inf \left\{ \frac{u' - k}{u'} + \frac{k}{u'} C^{\mathbb{Q}_T^*}(u', t') : k < u', t' \geq t \right\} \\ &\wedge \inf \left\{ \frac{u' - k}{u' - k'} A^C(k', t') + \frac{k - k'}{u' - k'} C^{\mathbb{Q}_T^*}(u', t'') : k' \leq k < u', t', t'' \geq t \right\} \\ &\wedge \inf \left\{ \frac{k' - k}{k'} + \frac{k}{k'} A^C(k', t') : 0 < k < k', t' \geq t \right\} \\ &\wedge \inf \left\{ \frac{k'' - k}{k'' - k'} A^C(k', t') + \frac{k - k'}{k'' - k'} A^C(k'', t'') : k' \leq k < k'', t', t'' \geq t \right\}. \end{aligned} \quad (\text{OB.3})$$

Note that $U^C(\cdot, \cdot; u)$ thus defined corresponds to the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{U}_T^u)$ for the choice

$$\mathbb{U}_T^u := \{(k, t, A^C(k, t)) : (k, t) \in \mathbb{K} \times \mathbb{T}\} \cup \{(u', t, C^{\mathbb{Q}_T^*}(u', t)) : u' \geq u, t \in \mathcal{T}\}. \quad (\text{OB.4})$$

Due to Lemma OA.3, $U^C(\cdot, \cdot; u)$ is associated to $\mathbb{Q}_T^u \in \mathcal{Q}(\mathcal{T})$ such that $C^{\mathbb{Q}_T^u}(k, t) = U^C(k, t; u)$ for all $(k, t) \in \Omega \times \mathcal{T}$. By construction, we moreover have that $\mathbb{Q}_T^u \in \mathcal{O}(\mathcal{T})$. In fact, as $C^{\mathbb{Q}_T^*}(\cdot, \cdot)$ corresponds to its own lower convex envelope, monotonicity yields that $B^C(k, t) \leq C^{\mathbb{Q}_T^*}(k, t) \leq U^C(k, t; u) \leq A^C(k, t)$ holds for all $(k, t) \in \mathbb{K} \times \mathbb{T}$.

With respect to equation (3.6), we have $U^C(\kappa, \tau; u) \leq U^C(\kappa, \tau)$ for any u . Letting $u \rightarrow \infty$, moreover observe that $U^C(\kappa, \tau; u) \rightarrow U^C(\kappa, \tau)$ monotonously. Hence, recalling that $C^{\mathbb{Q}_T^u}(\kappa, \tau) = U^C(\kappa, \tau; u)$, we have constructed a sequence $\{\mathbb{Q}_T^u\} \subset \mathcal{O}(\mathcal{T})$ that approaches the upper bound $U^C(\kappa, \tau)$ in equation (3.6), which must thus equal the supremum $\mathcal{U}^C(\kappa, \tau)$ in the call problem (3.4).

Call price, lower bound: Denoting the right-hand side in equation (3.7) by $L^C(\kappa, \tau)$, we analogously proceed in two steps. First, we show that $L^C(\kappa, \tau)$ is some lower bound for attainable call prices such that $C^{\mathbb{Q}_\tau}(\kappa, \tau) \geq L^C(\kappa, \tau)$ for all $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$. Second, we establish that $L^C(\kappa, \tau)$ coincides with the optimal lower bound (infimum) $\mathcal{L}^C(\kappa, \tau)$ in the call problem (3.4) by explicitly constructing a sequence $\{\mathbb{Q}_\tau^u\} \subset \mathcal{O}(\mathcal{T})$ with the property that $C^{\mathbb{Q}_\tau^u}(\kappa, \tau) \rightarrow L^C(\kappa, \tau)$ as $u \rightarrow \infty$. Since the case $\tau = 0$ is again straightforward, we only treat $\tau > 0$ in what follows.

To verify that $L^C(\kappa, \tau)$ is a lower bound, note that the convexity of $k \mapsto (1 - k)^+$ likewise yields for any $k', k'' \in \mathbb{K} \cup \{0, \infty\}$ with either $\kappa \leq k' < k''$ or $k' < k'' \leq \kappa$ the "sub-replicating" relation

$$(1 - \kappa)^+ \geq \frac{k'' - \kappa}{k'' - k'}(1 - k')^+ + \frac{\kappa - k'}{k'' - k'}(1 - k'')^+. \quad (\text{OB.5})$$

For infinite k'' , this specifically yields $(1 - \kappa)^+ \geq (1 - k')^+$ for all $k' \geq \kappa$. Every $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$ satisfies the constraints $C^{\mathbb{Q}_\tau}(k, \tau) \leq C^{\mathbb{Q}_\tau}(k, t) \leq A^C(k, t)$ for any $t \geq \tau$ and $C^{\mathbb{Q}_\tau}(k, \tau) \geq C^{\mathbb{Q}_\tau}(k, t) \geq B^C(k, t)$ for any $t \leq \tau$. Accounting for the signs of the respective fractions and using the convex order relation $C^{\mathbb{Q}_\tau}(\kappa, \tau) \geq C^{\mathbb{Q}_\tau}(\kappa, t)$ for any $t \leq \tau$, equation (OB.5) implies that for any $t' \leq t \leq t'', \tau$, the inequalities

$$\begin{aligned} C^{\mathbb{Q}_\tau}(\kappa, \tau) &\geq \frac{k'' - \kappa}{k'' - k'} C^{\mathbb{Q}_\tau}(k', t) + \frac{\kappa - k'}{k'' - k'} C^{\mathbb{Q}_\tau}(k'', t) \\ &\geq \begin{cases} \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') & , \text{ if } \kappa \leq k' < k'' \\ \frac{k'' - \kappa}{k'' - k'} A^C(k', t'') + \frac{\kappa - k'}{k'' - k'} B^C(k'', t') & , \text{ if } k' < k'' \leq \kappa \end{cases} \end{aligned} \quad (\text{OB.6})$$

hold for every $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$. Effectively, $L^C(\kappa, \tau)$ corresponds to the supremum of the right-hand side in equation (OB.6). Hence, we must have that $C^{\mathbb{Q}_\tau}(\kappa, \tau) \geq L^C(\kappa, \tau)$ for every $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$.

Using that $L^C(\kappa, \tau)$ is a lower bound, we next construct an approaching sequence of probability measures in $\mathcal{O}(\mathcal{T})$. For $t \in \mathcal{T}$ and some large $u > \kappa$, now define for $u' \geq u$ with $k', k'' \in \mathbb{K}$ and $t', t'' \in \mathbb{T}$

$$\begin{aligned} L^C(\kappa, t; u) &:= (1 - \kappa)^+ \vee \sup \left\{ \frac{u' - \kappa}{u' - k'} B^C(k', t') + \frac{\kappa - k'}{u' - k'} C^{\mathbb{Q}_\tau^*}(u', t'') : \kappa \leq k' < u', t' \leq t'', t \right\} \\ &\vee \sup \left\{ \frac{k' - \kappa}{k'} + \frac{\kappa}{k'} B^C(k', t') : 0 < k' \leq \kappa, t' \leq t \right\} \\ &\vee \sup \left\{ \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') : \kappa \leq k' < k'', t' \leq t'', t \right\} \\ &\vee \sup \left\{ \frac{k'' - \kappa}{k'' - k'} A^C(k', t'') + \frac{\kappa - k'}{k'' - k'} B^C(k'', t') : k' < k'' \leq \kappa, t' \leq t'', t \right\}. \end{aligned} \quad (\text{OB.7})$$

We may equivalently express equation (OB.7) for $k', k'' \in \mathbb{K} \cup [u, \infty)$ and $t' \in \mathbb{T}$ as

$$\begin{aligned} L^C(\kappa, t; u) &= (1 - \kappa)^+ \vee \sup \left\{ \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} \mathcal{E}(k'', t; \mathbb{U}_\tau^u) : \kappa \leq k' < k'', t' \leq t \right\} \\ &\vee \sup \left\{ \frac{k'' - \kappa}{k'' - k'} \mathcal{E}(k', t; \mathbb{U}_\tau^u) + \frac{\kappa - k'}{k'' - k'} B^C(k'', t') : k' < k'' \leq \kappa, t' \leq t \right\}. \end{aligned} \quad (\text{OB.8})$$

Construct the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{U}_\tau^u(\kappa))$ when setting

$$\begin{aligned} \mathbb{U}_\tau^u(\kappa) &:= \{(k, t, A^C(k, t)) : (k, t) \in \mathbb{K} \times \mathbb{T}\} \cup \{(\kappa, t, L^C(\kappa, t; u)) : t \in \mathcal{T}\} \\ &\cup \{(u', t, C^{\mathbb{Q}_\tau^*}(u', t)) : u' \geq u, t \in \mathcal{T}\}, \end{aligned} \quad (\text{OB.9})$$

which can be written as $\mathbb{L}_{\mathcal{T}}^u(\kappa) = \mathbb{U}_{\mathcal{T}}^u \cup \{L^C(\kappa, t; u) : t \in \mathcal{T}\}$. By construction, it follows that $\mathcal{E}(\kappa, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) \leq L^C(\kappa, t; u)$ for all $t \in \mathcal{T}$. Suppose that the inequality is strict at any given $t \in \mathcal{T}$, which implies that $\mathcal{E}(\kappa, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) = \mathcal{E}(\kappa, t; \mathbb{U}_{\mathcal{T}}^u)$ and may occur in two cases. First, using equation (OB.8), suppose that there exist admissible $\kappa \leq k' < k''$ and $t' \leq t$ with

$$\mathcal{E}(\kappa, t; \mathbb{U}_{\mathcal{T}}^u) < \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} \mathcal{E}(k'', t; \mathbb{U}_{\mathcal{T}}^u). \quad (\text{OB.10})$$

However, this would imply that $\frac{k'' - k'}{k'' - \kappa} \mathcal{E}(\kappa, t; \mathbb{U}_{\mathcal{T}}^u) + \frac{k' - \kappa}{k'' - \kappa} \mathcal{E}(k'', t; \mathbb{U}_{\mathcal{T}}^u) < B^C(k', t')$, in contradiction to the established correspondence of $\mathcal{E}(\cdot, \cdot; \mathbb{U}_{\mathcal{T}}^u)$ to a risk-neutral probability measure in $\mathcal{O}(\mathcal{T})$. Second, suppose that there exist admissible $k' < k'' \leq \kappa$ and $t' \leq t$ with

$$\mathcal{E}(\kappa, t; \mathbb{U}_{\mathcal{T}}^u) < \frac{k'' - \kappa}{k'' - k'} \mathcal{E}(k', t; \mathbb{U}_{\mathcal{T}}^u) + \frac{\kappa - k'}{k'' - k'} B^C(k'', t'). \quad (\text{OB.11})$$

Similarly, this would imply that $\frac{\kappa - k'}{\kappa - k'} \mathcal{E}(k', t; \mathbb{U}_{\mathcal{T}}^u) + \frac{k'' - k'}{\kappa - k'} \mathcal{E}(\kappa, t; \mathbb{U}_{\mathcal{T}}^u) < B^C(k'', t')$, leading to an analogous contradiction. Since equations (OB.10) and (OB.11) therefore cannot hold, we must have $\mathcal{E}(\kappa, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) = L^C(\kappa, t; u)$ for all $t \in \mathcal{T}$.

By Lemma OA.3, $\mathcal{E}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^u(\kappa))$ is associated to $\mathbb{Q}_{\mathcal{T}}^u \in \mathcal{Q}(\mathcal{T})$ such that $C^{\mathbb{Q}_{\mathcal{T}}^u}(k, t) = \mathcal{E}(k, t; \mathbb{L}_{\mathcal{T}}^u(\kappa))$ for all $(k, t) \in \Omega \times \mathcal{T}$. It remains to show that also $\mathbb{Q}_{\mathcal{T}}^u \in \mathcal{O}(\mathcal{T})$. The construction directly implies that $\mathcal{E}(k, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) \leq \mathcal{E}(k, t; \mathbb{U}_{\mathcal{T}}^u) \leq A^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. From the proof of the upper bound above, further recall that $\mathcal{E}(k, t; \mathbb{U}_{\mathcal{T}}^u) \geq B^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. To establish that also $\mathcal{E}(k, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) \geq B^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, note that violations can again happen in only two cases at any given $t \in \mathcal{T}$. First, suppose that there exist admissible $\kappa \leq k' < k''$ and $t' \leq t$ such that

$$\frac{k'' - k'}{k'' - \kappa} L^C(\kappa, t; u) + \frac{k' - \kappa}{k'' - \kappa} \mathcal{E}(k'', t; \mathbb{U}_{\mathcal{T}}^u) < B^C(k', t'). \quad (\text{OB.12})$$

However, this would imply that $\frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} \mathcal{E}(k'', t; \mathbb{U}_{\mathcal{T}}^u) > L^C(\kappa, t; u)$, contradicting the construction of $L^C(\kappa, t; u)$ in equation (OB.8). Second, suppose that there exist admissible $k' < k'' \leq \kappa$ and $t' \leq t$ such that

$$\frac{\kappa - k'}{\kappa - k'} \mathcal{E}(k', t; \mathbb{U}_{\mathcal{T}}^u) + \frac{k'' - k'}{\kappa - k'} L^C(\kappa, t; u) < B^C(k'', t'). \quad (\text{OB.13})$$

However, then $\frac{k'' - \kappa}{k'' - k'} \mathcal{E}(k', t; \mathbb{U}_{\mathcal{T}}^u) + \frac{\kappa - k'}{k'' - k'} B^C(k'', t') > L^C(\kappa, t; u)$, likewise contradicting the construction of $L^C(\kappa, t; u)$. Since equations (OB.12) and (OB.13) thus cannot hold as stated, it must be the case that $\mathcal{E}(k, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) \geq B^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. In conclusion, we have $\mathbb{Q}_{\mathcal{T}}^u \in \mathcal{O}(\mathcal{T})$.

With regards to equation (3.7), it holds that $L^C(\kappa, \tau; u) \geq L^C(\kappa, \tau)$ for each u with $L^C(\kappa, \tau; u) \rightarrow L^C(\kappa, \tau)$ monotonously as $u \rightarrow \infty$. Recalling that specifically $C^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, \tau) = L^C(\kappa, \tau; u)$, we have constructed a sequence $\{\mathbb{Q}_{\mathcal{T}}^u\} \subset \mathcal{O}(\mathcal{T})$ that approaches the lower bound $L^C(\kappa, \tau)$ in equation (3.7), which must thus equal the infimum $\mathcal{L}^C(\kappa, \tau)$ in the call problem (3.4).

OB.2 Proof of Proposition 3.2

In order to establish the proposition, we first extend the construction in equation (OA.5) and define on $\Omega \times \mathcal{T}$ the maturity-interpolated lower convex envelope by linear interpolation between the straddling maturities $t', t'' \in \mathbb{T} \cup \{\infty\}$ with $t' \leq \tau < t''$, such that

$$\bar{\mathcal{E}}(\kappa, \tau; \mathbb{C}) := \frac{t'' - \tau}{t'' - t'} \mathcal{E}(\kappa, t'; \mathbb{C}) + \frac{\tau - t'}{t'' - t'} \mathcal{E}(\kappa, t''; \mathbb{C}). \quad (\text{OB.14})$$

By construction, $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{C})$ preserves the relevant properties in Lemma OA.2. Under the conditions of Lemma OA.3, $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{C})$ therefore defines an arbitrage-free option price surface $C^{\bar{\mathbb{Q}}_\tau}$ on $\Omega \times \mathcal{T}$ for $\bar{\mathbb{Q}}_\tau \in \bar{\mathcal{Q}}(\mathcal{T})$. In particular, for any $\tau \in \mathcal{T}$, it holds that

$$\bar{\mathbb{Q}}_\tau := \frac{t'' - \tau}{t'' - t'} \mathbb{Q}_{t'} + \frac{\tau - t'}{t'' - t'} \mathbb{Q}_{t''}, \quad (\text{OB.15})$$

where $\mathbb{Q}_\tau \in \mathcal{Q}(\mathcal{T})$ is the probability measure associated to $\mathcal{E}(\cdot, \cdot; \mathbb{C})$.

With this extended construction, we are now in the position to prove the proposition. The arguments closely follow the proof of Proposition 3.1. We again focus on the bounds for calls. The corresponding bounds for puts then follow from the put-call parity relations (3.5).

Call price, upper bound: Denote the right-hand side in equation (3.13) by $\bar{U}^C(\kappa, \tau)$. Clearly, it holds that $C^{\bar{\mathbb{Q}}_\tau}(\kappa, \tau) \leq \bar{U}^C(\kappa, \tau)$ for all $\bar{\mathbb{Q}}_\tau \in \bar{\mathcal{O}}(\mathcal{T})$. Hence, $\bar{U}^C(\kappa, \tau)$ is an upper bound over $\bar{\mathcal{O}}(\mathcal{T})$.

To establish optimality of this bound, take $\mathbb{U}_\tau^u$ as in equation (OB.4) and consider the maturity-interpolated lower convex envelope $\bar{\mathcal{E}}(\cdot, \cdot; u) := \bar{\mathcal{E}}(\cdot, \cdot; \mathbb{U}_\tau^u)$. As noted above, $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{U}_\tau^u)$ is associated to some $\bar{\mathbb{Q}}_\tau^u \in \bar{\mathcal{Q}}(\mathcal{T})$. Since $\bar{\mathcal{E}}(\cdot, t; \mathbb{U}_\tau^u) = \mathcal{E}(\cdot, t; \mathbb{U}_\tau^u)$ for any $t \in \mathbb{T}$, it moreover holds that $\bar{\mathbb{Q}}_\tau^u \in \bar{\mathcal{O}}(\mathcal{T})$.

Concretely, for the straddling maturities $t', t'' \in \mathbb{T} \cup \{\infty\}$ with $t' \leq \tau < t''$, we may write

$$\bar{U}^C(\kappa, \tau; u) = \frac{t'' - \tau}{t'' - t'} U^C(\kappa, t'; u) + \frac{\tau - t'}{t'' - t'} U^C(\kappa, t''; u)$$

in terms of $U^C(\kappa, t; u)$ as in equation (OB.3). From the proof of Proposition 3.1, we have that $U^C(\kappa, t'; u) \rightarrow U^C(\kappa, t') = \mathcal{U}^C(\kappa, t')$ and $U^C(\kappa, t''; u) \rightarrow U^C(\kappa, t'') = \mathcal{U}^C(\kappa, t'')$ as $u \rightarrow \infty$. Therefore, it follows that also $\bar{U}^C(\kappa, \tau; u) \rightarrow \bar{U}^C(\kappa, \tau)$ as $u \rightarrow \infty$. Since $C^{\bar{\mathbb{Q}}_\tau^u}(\kappa, \tau) = \bar{U}^C(\kappa, \tau; u)$, we thus have found a sequence $\{\bar{\mathbb{Q}}_\tau^u\} \subset \bar{\mathcal{O}}(\mathcal{T})$ that approaches the upper bound $\bar{U}^C(\kappa, \tau)$, which must thus equal the supremum $\bar{U}^C(\kappa, \tau)$ in the call problem (3.4) over $\bar{\mathcal{O}}(\mathcal{T})$.

Call price, lower bound: Denote the right-hand side in equation (3.14) by $\bar{L}^C(\kappa, \tau)$. We immediately have that $C^{\bar{\mathbb{Q}}_\tau}(\kappa, \tau) \geq \bar{L}^C(\kappa, \tau)$ for all $\bar{\mathbb{Q}}_\tau \in \bar{\mathcal{O}}(\mathcal{T})$, so that $\bar{L}^C(\kappa, \tau)$ is a lower bound over $\bar{\mathcal{O}}(\mathcal{T})$.

In order to show its optimality, consider the maturity-interpolated lower convex envelope $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{L}_\tau^u(\kappa))$, using $\mathbb{L}_\tau^u(\kappa)$ as in equation (OB.9), which by construction is associated to some $\bar{\mathbb{Q}}_\tau^u \in \bar{\mathcal{Q}}(\mathcal{T})$. As $\bar{\mathcal{E}}(\cdot, t; \mathbb{L}_\tau^u(\kappa)) = \mathcal{E}(\cdot, t; \mathbb{L}_\tau^u(\kappa))$ for all $t \in \mathbb{T}$, we further have that $\bar{\mathbb{Q}}_\tau^u \in \bar{\mathcal{O}}(\mathcal{T})$.

It is shown in the proof of Proposition 3.1 that $\mathcal{E}(\kappa, t; \mathbb{L}_\tau^u(\kappa)) = L^C(\kappa, t; u)$ for all $t \in \mathcal{T}$, where the latter is defined in equation (OB.8). Analogously, we may conclude that $\bar{\mathcal{E}}(\kappa, t; \mathbb{L}_\tau^u(\kappa)) = \bar{L}^C(\kappa, t; u)$ for all $t \in \mathcal{T}$. Given the straddling maturities $t', t'' \in \mathbb{T} \cup \{\infty\}$ with $t' \leq \tau < t''$, we thus have the relation

$$\bar{L}^C(\kappa, \tau; u) = \frac{t'' - \tau}{t'' - t'} L^C(\kappa, t'; u) + \frac{\tau - t'}{t'' - t'} L^C(\kappa, t''; u).$$

As established in the proof of Proposition 3.1, it holds that $L^C(\kappa, t'; u) \rightarrow L^C(\kappa, t') = \mathcal{L}^C(\kappa, t')$ and $L^C(\kappa, t''; u) \rightarrow L^C(\kappa, t'') = \mathcal{L}^C(\kappa, t'')$ as $u \rightarrow \infty$. We thereby obtain that $\bar{L}^C(\kappa, \tau; u) \rightarrow \bar{L}^C(\kappa, \tau)$ as $u \rightarrow \infty$. Due to the identity $C^{\bar{\mathbb{Q}}_\tau^u}(\kappa, \tau) = \bar{L}^C(\kappa, \tau; u)$, the sequence $\{\bar{\mathbb{Q}}_\tau^u\} \subset \bar{\mathcal{O}}(\mathcal{T})$ approaches the lower bound $\bar{L}^C(\kappa, \tau)$, which must thus equal the infimum $\bar{L}^C(\kappa, \tau)$ in the call problem (3.4) over $\bar{\mathcal{O}}(\mathcal{T})$.

OB.3 Proof of Proposition 3.3

Similarly as in the proof of Proposition 3.1, the proof for each case proceeds in two steps. First, we establish a concrete upper or lower bound across all admissible $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$. Second, we explicitly construct a sequence $\{\mathbb{Q}_\tau^u\} \subset \mathcal{O}(\mathcal{T})$ of admissible probability measures, parameterized by u (and possibly another

parameter), that approaches the constructed upper or lower bound as $u \rightarrow \infty$. As a consequence, the upper or lower bounds must correspond to the supremum or infimum, respectively. Since the case $\tau = 0$ is trivial, we will focus the arguments on $\tau > 0$.

Left-tail probability, upper bound: Denote the right-hand side in equation (3.22) by $U^{LTP}(\kappa, \tau)$. Given any $k' > \kappa$, we have the "super-replicating" relation

$$\mathbb{I}_{[0, \kappa]}(k) \leq 1 + \frac{(k - k')^+ - (k - \kappa)^+}{k' - \kappa}. \quad (\text{OB.16})$$

Moreover, the trivial bound $\mathbb{I}_{[0, \kappa]}(k) \leq 1$ holds, essentially corresponding to the limit $k' \rightarrow \infty$ in equation (OB.16). By construction of the option price bounds in Proposition 3.1, every feasible $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$ specifically satisfies $\mathcal{L}^C(k, \tau) \leq C^{\mathbb{Q}_\tau}(k, \tau) \leq \mathcal{U}^C(k, \tau)$ for each $k \in \mathbb{K}$. Therefore, $U^{LTP}(\kappa, \tau)$ is an upper bound, since equation (OB.16) yields for $k' \in \mathbb{K}$ that

$$LTP^{\mathbb{Q}_\tau}(\kappa, \tau) \leq U^{LTP}(\kappa, \tau) := 1 \wedge \inf \left\{ 1 + \frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa, \tau)}{k' - \kappa} : \kappa < k' \right\}, \quad (\text{OB.17})$$

which by the put-call parity relations (3.5) corresponds to the expression stated in equation (3.22).

Take some large $u > \kappa$ and define $U^C(k, \tau; u)$ and $L^C(k, \tau; u)$ for $(k, \tau) \in \Omega \times \mathcal{T}$ as in equations (OB.3) and (OB.8), respectively, for an arbitrary $\mathbb{Q}_\tau^* \in \mathcal{O}(\mathcal{T})$. Construct the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{L}_\tau^u(\kappa))$ when setting $\mathbb{L}_\tau^u(\kappa)$ according to equation (OB.9), which by Lemma OA.3 and the reasoning in the proof of Proposition 3.1 is associated to a probability measure $\mathbb{Q}_\tau^u \in \mathcal{O}(\mathcal{T})$. Observe that by construction of the lower convex envelope, which satisfies $\mathcal{E}(\kappa, \tau; \mathbb{L}_\tau^u(\kappa)) = L^C(\kappa, \tau; u)$, we have that

$$\begin{aligned} LTP^{\mathbb{Q}_\tau^u}(\kappa, \tau) &= 1 + \partial_{k+} \mathcal{E}(\kappa, \tau; \mathbb{L}_\tau^u(\kappa)) \\ &= \inf \left\{ 1 + \frac{U^C(k', \tau; u) - L^C(\kappa, \tau; u)}{k' - \kappa} : \kappa < k' \right\}. \end{aligned} \quad (\text{OB.18})$$

To show that $LTP^{\mathbb{Q}_\tau^u}(\kappa, \tau) \rightarrow U^{LTP}(\kappa, \tau)$ as $u \rightarrow \infty$, we invoke Sion's Minimax Theorem (cf. Corollary 3.3 in Sion (1958)). For this, after a change of variables $\tilde{u} := 1/u$, consider the function

$$F(k, \tilde{u}) := \tilde{F}(k, \tilde{u}) \wedge 0 := \frac{U^C(k, \tau; 1/\tilde{u}) - L^C(\kappa, \tau; 1/\tilde{u})}{k - \kappa} \wedge 0,$$

defined on $(\kappa, \infty) \times [0, M]$ for some small $0 < M < 1/\kappa$, using at $\tilde{u} = 0$ the continuous extensions $U^C(k, \tau; 0) := \mathcal{U}^C(k, \tau)$ and $L^C(k, \tau; 0) := \mathcal{L}^C(k, \tau)$ in terms of the option price bounds in Proposition 3.1. Observe that for any given $k > \kappa$, $F(k, \cdot)$ is continuous and monotonous, since the convergences $U^C(k, \tau; 1/\tilde{u}) \rightarrow \mathcal{U}^C(k, \tau)$ and $L^C(k, \tau; 1/\tilde{u}) \rightarrow \mathcal{L}^C(k, \tau)$ are monotonous as $\tilde{u} \rightarrow 0$. The latter implies that $F(k, \cdot)$ is quasi-concave. Moreover, for fixed $0 \leq \tilde{u} \leq M$, $F(\cdot, \tilde{u})$ is continuous and quasi-convex, because $\tilde{F}(\cdot, \tilde{u})$ is convex. As a consequence, Sion's Minimax Theorem applies to the function F and we indeed

obtain that

$$\begin{aligned}
& \lim_{u \rightarrow \infty} \inf \left\{ \frac{U^C(k', \tau; u) - L^C(\kappa, \tau; u)}{k' - \kappa} : \kappa < k' \right\} \\
&= \sup \left\{ \inf \left\{ \frac{U^C(k', \tau; 1/\tilde{u}) - L^C(\kappa, \tau; 1/\tilde{u})}{k' - \kappa} \wedge 0 : \kappa < k' \right\} : 0 \leq \tilde{u} \leq M \right\} \\
&= \inf \left\{ \sup \left\{ \frac{U^C(k', \tau; 1/\tilde{u}) - L^C(\kappa, \tau; 1/\tilde{u})}{k' - \kappa} \wedge 0 : 0 \leq \tilde{u} \leq M \right\} : \kappa < k' \right\} \\
&= \inf \left\{ \lim_{u \rightarrow \infty} \frac{U^C(k', \tau; u) - L^C(\kappa, \tau; u)}{k' - \kappa} \wedge 0 : \kappa < k' \right\} \\
&= \inf \left\{ \frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa, \tau)}{k' - \kappa} \wedge 0 : \kappa < k' \right\},
\end{aligned}$$

where the last equality uses that as $u \rightarrow \infty$,

$$\frac{U^C(k', \tau; u) - L^C(\kappa, \tau; u)}{k' - \kappa} \wedge 0 \longrightarrow \frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa, \tau)}{k' - \kappa} \wedge 0.$$

Observe that the infimum in the final equality is equivalent to $U^{LTP}(\kappa, \tau) - 1$.

With the preceding construction, we have found a sequence $\{\mathbb{Q}_{\mathcal{T}}^u\} \subset \mathcal{O}(\mathcal{T})$ that approaches the upper bound $U^{LTP}(\kappa, \tau)$ in equation (OB.17) as $u \rightarrow \infty$, and thus the equivalent expression in equation (3.22), which therefore must coincide with the supremum $\mathcal{U}^{LTP}(\kappa, \tau)$ in the left-tail problem (3.21). In order to determine $\mathcal{U}^{LTP}(\kappa, \tau)$, it suffices to consider the infimum over $k' \in \mathbb{K}$ whenever $\tau > 0$. The same expression also holds in the case $\tau = 0$ for $k' \in \{1\}$.

Left-tail probability, lower bound: Denote the right-hand side in equation (3.23) by $L^{LTP}(\kappa, \tau)$. Any $k' \leq \kappa$ now yields the "sub-replicating" relation

$$\mathbb{1}_{[0, \kappa]}(k) \geq 1 + \lim_{\tilde{k} \downarrow \kappa} \frac{(k - \tilde{k})^+ - (k - k')^+}{\tilde{k} - k'}. \quad (\text{OB.19})$$

Equivalently, for $k' < \kappa$, this relation may be written as

$$\mathbb{1}_{[0, \kappa]}(k) \geq 1 + \frac{(k - \kappa)^+ - (k - k')^+}{\kappa - k'}, \quad \mathbb{1}_{[0, \kappa]}(k) = 1 + \partial_{k+}(k - \kappa)^+ = 1 + \lim_{\tilde{k} \downarrow \kappa} \frac{(k - \tilde{k})^+ - (k - \kappa)^+}{\tilde{k} - \kappa}.$$

Hence, for every feasible $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$, equation (OB.19) implies that $L^{LTP}(\kappa, \tau)$ is a lower bound. Concretely, we have for $k' \in \mathbb{K} \cup \{0, \kappa\}$ that

$$LTP^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau) \geq L^{LTP}(\kappa, \tau) := \sup \left\{ 1 + \lim_{\eta \downarrow 0} \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k', \tau)}{\kappa + \eta - k'} : k' \leq \kappa \right\}, \quad (\text{OB.20})$$

which by the put-call parity relations (3.5) corresponds to the expression stated in equation (3.23).

Taking $\mathbb{L}_{\mathcal{T}}^u(\kappa + \eta)$ for some large $u > \kappa$ and small $\eta > 0$ now leads to the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^u(\kappa + \eta))$, for which it holds by construction that $\mathcal{E}(\kappa + \eta, \tau; \mathbb{L}_{\mathcal{T}}^u(\kappa + \eta)) = L^C(\kappa + \eta, \tau; u)$. The associated probability measure $\mathbb{Q}_{\mathcal{T}}^{u, \eta} \in \mathcal{O}(\mathcal{T})$ satisfies

$$\begin{aligned}
LTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) &= 1 + \partial_{k+} \mathcal{E}(\kappa, \tau; \mathbb{L}_{\mathcal{T}}^u(\kappa + \eta)) \\
&\leq \sup \left\{ 1 + \frac{L^C(\kappa + \eta, \tau; u) - U^C(k', \tau; u)}{\kappa + \eta - k'} : k' < \kappa + \eta \right\}. \quad (\text{OB.21})
\end{aligned}$$

In order to show that $LTP^{\mathbb{Q}_T^{u,\eta}}(\kappa, \tau) \rightarrow L^{LTP}(\kappa, \tau)$ as $u \rightarrow \infty$ and $\eta \rightarrow 0$ jointly, it suffices to verify that the right-hand side of equation (OB.21) converges to $L^{LTP}(\kappa, \tau)$. In this regard, note that the limits may be taken sequentially, in any order, as the right-hand side of equation (OB.21) is monotone in u and η . First taking the limit $u \rightarrow \infty$, we invoke Sion's Minimax Theorem for fixed $\eta > 0$ for the function

$$F(k, \tilde{u}) := \tilde{F}(k, \tilde{u}) \vee (-1) := \frac{L^C(\kappa + \eta, \tau; 1/\tilde{u}) - U^C(k, \tau; 1/\tilde{u})}{\kappa + \eta - k} \vee (-1),$$

defined on $[0, \kappa + \eta] \times [0, M]$ for some small $0 < M < 1/\kappa$. Here, we likewise use at $\tilde{u} = 0$ the continuous extensions $U^C(k, \tau; 0) := \mathcal{U}^C(k, \tau)$ and $L^C(k, \tau; 0) := \mathcal{L}^C(k, \tau)$ in terms of the option price bounds in Proposition 3.1. For any given $0 \leq k < \kappa + \eta$, $F(k, \cdot)$ is continuous and monotonous, hence quasi-convex. Moreover, for any given $0 \leq \tilde{u} \leq M$, $F(\cdot, \tilde{u})$ is continuous and quasi-concave, due to $\tilde{F}(\cdot, \tilde{u})$ being concave. Sion's Minimax Theorem applied to F therefore yields for any $\eta > 0$ that

$$\begin{aligned} & \lim_{u \rightarrow \infty} \sup \left\{ \frac{L^C(\kappa + \eta, \tau; u) - U^C(k', \tau; u)}{\kappa + \eta - k'} \vee (-1) : k' < \kappa + \eta \right\} \\ &= \inf \left\{ \sup \left\{ \frac{L^C(\kappa + \eta, \tau; 1/\tilde{u}) - U^C(k', \tau; 1/\tilde{u})}{\kappa + \eta - k'} \vee (-1) : k' < \kappa + \eta \right\} : 0 \leq \tilde{u} \leq M \right\} \\ &= \sup \left\{ \inf \left\{ \frac{L^C(\kappa + \eta, \tau; 1/\tilde{u}) - U^C(k', \tau; 1/\tilde{u})}{\kappa + \eta - k'} \vee (-1) : 0 \leq \tilde{u} \leq M \right\} : k' < \kappa + \eta \right\} \\ &= \sup \left\{ \lim_{u \rightarrow \infty} \frac{L^C(\kappa + \eta, \tau; u) - U^C(k', \tau; u)}{\kappa + \eta - k'} \vee (-1) : k' < \kappa + \eta \right\} \\ &= \sup \left\{ \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k', \tau)}{\kappa + \eta - k'} \vee (-1) : k' < \kappa + \eta \right\}, \end{aligned}$$

where the final equality uses that as $u \rightarrow \infty$, we have for any $\eta > 0$ that

$$\frac{L^C(\kappa + \eta, \tau; u) - U^C(k', \tau; u)}{\kappa + \eta - k'} \vee (-1) \rightarrow \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k', \tau)}{\kappa + \eta - k'} \vee (-1).$$

Next taking the limit as $\eta \rightarrow 0$, another application of Sion's Minimax Theorem considers the function

$$G(k, \eta) := \tilde{G}(k, \eta) \vee (-1) := \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k \wedge (\kappa + \eta), \tau)}{\kappa + \eta - k \wedge (\kappa + \eta)} \vee (-1),$$

defined on $[0, k + \delta] \times [0, \delta]$ for some small $\delta > 0$, where the value $\tilde{G}(k, \eta) = \tilde{G}(\kappa + \eta, \eta)$ for $k \geq \kappa + \eta$ is determined via continuous extension. For any given $0 \leq k < \kappa + \delta$, $G(k, \cdot)$ is continuous and monotonous, hence quasi-convex. Moreover, for any given $0 \leq \eta \leq \delta$, $G(\cdot, \eta)$ is continuous and quasi-concave, as $\tilde{G}(\cdot, \eta)$ is concave. Sion's Minimax Theorem applied to G thus gives

$$\begin{aligned} & \lim_{\eta \downarrow 0} \sup \left\{ \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k', \tau)}{\kappa + \eta - k'} \vee (-1) : k' < \kappa + \eta \right\} \\ &= \inf \left\{ \sup \left\{ \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k' \wedge (\kappa + \eta), \tau)}{\kappa + \eta - k' \wedge (\kappa + \eta)} \vee (-1) : k' < \kappa + \delta \right\} : 0 \leq \eta \leq \delta \right\} \\ &= \sup \left\{ \inf \left\{ \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k' \wedge (\kappa + \eta), \tau)}{\kappa + \eta - k' \wedge (\kappa + \eta)} \vee (-1) : 0 \leq \eta \leq \delta \right\} : k' < \kappa + \delta \right\} \\ &= \sup \left\{ \lim_{\eta \downarrow 0} \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k', \tau)}{\kappa + \eta - k'} \vee (-1) : k' \leq \kappa \right\}, \end{aligned}$$

using that as $\eta \rightarrow 0$,

$$\frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k' \wedge (\kappa + \eta), \tau)}{\kappa + \eta - k' \wedge (\kappa + \eta)} \vee (-1) \longrightarrow \lim_{\eta \downarrow 0} \frac{\mathcal{L}^C(\kappa + \eta, \tau) - \mathcal{U}^C(k', \tau)}{\kappa + \eta - k'} \vee (-1).$$

Note that the supremum in the final equality is indeed equivalent to $L^{LTP}(\kappa, \tau) - 1$, as the choice $k' = 0$ automatically yields a value ≥ -1 .

The constructed sequence $\{\mathbb{Q}_\tau^{u, \eta}\} \subset \mathcal{O}(\mathcal{T})$ thus approaches the lower bound $L^{LTP}(\kappa, \tau)$ in equation (OB.20) as $u \rightarrow \infty$ and $\eta \rightarrow 0$ jointly, and thereby also the equivalent expression in equation (3.23). As a consequence, these expressions must coincide with the infimum $\mathcal{L}^{LTP}(\kappa, \tau)$ in the left-tail problem (3.21). To determine $\mathcal{L}^{LTP}(\kappa, \tau)$, it suffices to consider the supremum over $k' \in \mathbb{K} \cup \{0, \kappa\}$ if $\tau > 0$. The same expression also holds in the case $\tau = 0$ for $k' \in \{0, 1\}$.

Right-tail probability, upper bound: The argument follows the one for the upper bound of left-tail probabilities. Denote the right-hand side in equation (3.24) by $U^{RTP}(\kappa, \tau)$. For any $k' < \kappa$, note the "super-replicating" relation

$$\mathbb{I}_{[\kappa, \infty)}(k) \leq \frac{(k - k')^+ - (k - \kappa)^+}{k' - \kappa}. \quad (\text{OB.22})$$

Using the option price bounds in Proposition 3.1 and equation (OB.22), every feasible solution $\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})$ thus satisfies

$$RTP^{\mathbb{Q}_\tau}(\kappa, \tau) \leq U^{RTP}(\kappa, \tau) := \inf \left\{ -\frac{\mathcal{L}^C(\kappa, \tau) - \mathcal{U}^C(k', \tau)}{\kappa - k'} : k' < \kappa \right\} \quad (\text{OB.23})$$

for $k' \in \mathbb{K} \cup \{0\}$, so that $U^{RTP}(\kappa, \tau)$ is an upper bound.

As before, take $\mathbb{L}_\tau^u(\kappa)$ according to equation (OB.9) to construct a probability measure $\mathbb{Q}_\tau^u \in \mathcal{Q}(\mathcal{T})$. We thereby obtain

$$\begin{aligned} RTP^{\mathbb{Q}_\tau^u}(\kappa, \tau) &= -\partial_{k-} \mathcal{E}(\kappa, \tau; \mathbb{L}_\tau^u(\kappa)) \\ &= \inf \left\{ -\frac{L^C(\kappa, \tau; u) - U^C(k', \tau; u)}{\kappa - k'} : k' < \kappa \right\}. \end{aligned} \quad (\text{OB.24})$$

To show that $RTP^{\mathbb{Q}_\tau^u}(\kappa, \tau) \rightarrow U^{RTP}(\kappa, \tau)$ as $u \rightarrow \infty$, we once more apply Sion's Minimax Theorem. In the present case, we consider the function

$$F(k, \tilde{u}) := -\frac{L^C(\kappa, \tau; 1/\tilde{u}) - U^C(k, \tau; 1/\tilde{u})}{\kappa - k} \wedge 1,$$

defined on $[0, k'] \times [0, M]$ for some small $0 < M < 1/\kappa$. After verifying the applicability of the theorem to F using analogous arguments as before, we thereby get

$$\begin{aligned} &\lim_{u \rightarrow \infty} \inf \left\{ -\frac{L^C(\kappa, \tau; u) - U^C(k', \tau; u)}{\kappa - k'} \wedge 1 : k' < \kappa \right\} \\ &= \sup \left\{ \inf \left\{ -\frac{L^C(\kappa, \tau; 1/\tilde{u}) - U^C(k', \tau; 1/\tilde{u})}{\kappa - k'} \wedge 1 : k' < \kappa \right\} : 0 \leq \tilde{u} \leq M \right\} \\ &= \inf \left\{ \sup \left\{ -\frac{L^C(\kappa, \tau; 1/\tilde{u}) - U^C(k', \tau; 1/\tilde{u})}{\kappa - k'} \wedge 1 : 0 \leq \tilde{u} \leq M \right\} : k' < \kappa \right\} \\ &= \inf \left\{ \lim_{u \rightarrow \infty} -\frac{L^C(\kappa, \tau; u) - U^C(k', \tau; u)}{\kappa - k'} \wedge 1 : k' < \kappa \right\} \\ &= \inf \left\{ -\frac{\mathcal{L}^C(\kappa, \tau) - \mathcal{U}^C(k', \tau)}{\kappa - k'} \wedge 1 : k' < \kappa \right\}, \end{aligned}$$

using that

$$-\frac{L^C(\kappa, \tau; u) - U^C(k', \tau; u)}{\kappa - k'} \wedge 1 \longrightarrow -\frac{\mathcal{L}^C(\kappa, \tau) - \mathcal{U}^C(k', \tau)}{\kappa - k'} \wedge 1.$$

The infimum in the final equality is equivalent to $U^{RTP}(\kappa, \tau)$, since $k' = 0$ always yields a value ≤ 1 .

We therefore have constructed a sequence $\{\mathbb{Q}_{\mathcal{T}}^u\} \subset \mathcal{O}(\mathcal{T})$ that approaches the upper bound $U^{RTP}(\kappa, \tau)$ in equation (OB.23) as $u \rightarrow \infty$, coinciding with the expression in equation (3.24). These expressions thus equal the supremum $\mathcal{U}^{RTP}(\kappa, \tau)$ in the right-tail problem (3.21). To determine $\mathcal{U}^{RTP}(\kappa, \tau)$, it now suffices to consider the infimum over $k' \in \mathbb{K} \cup \{0\}$ if $\tau > 0$, while using $k' \in \{0, 1\}$ if $\tau = 0$.

Right-tail probability, lower bound: The argument follows the one for the lower bound of left-tail probabilities. Denote the right-hand side in equation (3.25) by $L^{RTP}(\kappa, \tau)$. For any $k' \geq \kappa$, we have the "sub-replicating" relation

$$\mathbb{I}_{[\kappa, \infty)}(k) \geq \lim_{\tilde{k} \uparrow \kappa} -\frac{(k - k')^+ - (k - \tilde{k})^+}{k' - \tilde{k}}. \quad (\text{OB.25})$$

For any $k' > \kappa$, this is equivalent to

$$\mathbb{I}_{[\kappa, \infty)}(k) \geq -\frac{(k - k')^+ - (k - \kappa)^+}{k' - \kappa}, \quad \mathbb{I}_{[\kappa, \infty)}(k) = -\partial_{k-}(k - \kappa)^+ = \lim_{\tilde{k} \uparrow \kappa} -\frac{(k - \kappa)^+ - (k - \tilde{k})^+}{\kappa - \tilde{k}}.$$

Trivially, it further holds that $\mathbb{I}_{[\kappa, \infty)}(k) \geq 0$, essentially when $k' \rightarrow \infty$ in equation (OB.25). For every feasible solution $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$, from equation (OB.25) we therefore have that

$$RTP^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau) \geq L^{RTP}(\kappa, \tau) := 0 \vee \sup \left\{ \lim_{\eta \downarrow 0} -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' - (\kappa - \eta)} : \kappa \leq k' \right\} \quad (\text{OB.26})$$

for $k' \in \mathbb{K}$, so that $L^{RTP}(\kappa, \tau)$ is a lower bound. If necessary, we may here employ the consistent extension $\mathcal{L}^C(k, \tau) = 1 - k$ for any $k < 0$.

Now take $\mathbb{L}_{\mathcal{T}}^u(\kappa - \eta)$ for some small $\eta > 0$ to construct a probability measure $\mathbb{Q}_{\mathcal{T}}^{u, \eta} \in \mathcal{Q}(\mathcal{T})$. Likewise, $L^C(k, \tau; u) = 1 - k$ yields a consistent extension for any $k < 0$. For $k' \in \mathbb{K} \cup \{u\}$, it holds that

$$\begin{aligned} RTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) &= -\partial_{k-}\mathcal{E}(\kappa, \tau; \mathbb{L}_{\mathcal{T}}^u(\kappa - \eta)) \\ &\leq \sup \left\{ -\frac{U^C(k', \tau; u) - L^C(\kappa - \eta, \tau; u)}{k' - (\kappa - \eta)} : \kappa - \eta < k' \right\}. \end{aligned} \quad (\text{OB.27})$$

To show that $RTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) \rightarrow L^{RTP}(\kappa, \tau)$ as $u \rightarrow \infty$ and $\eta \rightarrow 0$ jointly, it suffices to establish the convergence of the right-hand side of equation (OB.27) to $L^{RTP}(\kappa, \tau)$. We may again take limits sequentially due to monotonicity. For the limit as $u \rightarrow \infty$, given any $\eta > 0$, we apply Sion's Minimax Theorem to the function

$$F(k, \tilde{u}) := -\frac{U^C(k, \tau; 1/\tilde{u}) - L^C(\kappa - \eta, \tau; 1/\tilde{u})}{k - (\kappa - \eta)} \vee 0,$$

defined on $(\kappa - \eta, \infty) \times [0, M]$ for some small $0 < M < 1/\kappa$. By analogous arguments as before, the

theorem applies to F and yields

$$\begin{aligned}
& \lim_{u \rightarrow \infty} \sup \left\{ -\frac{U^C(k', \tau; u) - L^C(\kappa - \eta, \tau; u)}{k' - (\kappa - \eta)} \vee 0 : \kappa - \eta < k' \right\} \\
&= \inf \left\{ \sup \left\{ -\frac{U^C(k', \tau; 1/\tilde{u}) - L^C(\kappa - \eta, \tau; 1/\tilde{u})}{k' - (\kappa - \eta)} \vee 0 : \kappa - \eta < k' \right\} : 0 \leq \tilde{u} \leq M \right\} \\
&= \sup \left\{ \inf \left\{ -\frac{U^C(k', \tau; 1/\tilde{u}) - L^C(\kappa - \eta, \tau; 1/\tilde{u})}{k' - (\kappa - \eta)} \vee 0 : 0 \leq \tilde{u} \leq M \right\} : \kappa - \eta < k' \right\} \\
&= \sup \left\{ \lim_{u \rightarrow \infty} -\frac{U^C(k', \tau; u) - L^C(\kappa - \eta, \tau; u)}{k' - (\kappa - \eta)} \vee 0 : \kappa - \eta < k' \right\} \\
&= \sup \left\{ -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' - (\kappa - \eta)} \vee 0 : \kappa - \eta < k' \right\},
\end{aligned}$$

using that

$$-\frac{U^C(k', \tau; u) - L^C(\kappa - \eta, \tau; u)}{k' - (\kappa - \eta)} \vee 0 \longrightarrow -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa, \tau)}{k' - \kappa} \vee 0.$$

Another application of Sion's Minimax Theorem for the limit $\eta \rightarrow 0$ is based on the function

$$G(k, \eta) := -\frac{\mathcal{U}^C(k \vee (\kappa - \eta), \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k \vee (\kappa - \eta) - (\kappa - \eta)} \vee 0,$$

defined on $(k - \eta, \infty) \times [0, \delta]$ for some small $\delta > 0$, using appropriate continuous extension. Analogous arguments as before verify the applicability of the theorem to G and thereby justify that

$$\begin{aligned}
& \lim_{\eta \downarrow 0} \sup \left\{ -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' - (\kappa - \eta)} \vee 0 : \kappa - \eta < k' \right\} \\
&= \inf \left\{ \sup \left\{ -\frac{\mathcal{U}^C(k' \vee (\kappa - \eta), \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' \vee (\kappa - \eta) - (\kappa - \eta)} \vee 0 : \kappa - \delta < k' \right\} : 0 \leq \eta \leq \delta \right\} \\
&= \sup \left\{ \inf \left\{ -\frac{\mathcal{U}^C(k' \vee (\kappa - \eta), \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' \vee (\kappa - \eta) - (\kappa - \eta)} \vee 0 : 0 \leq \eta \leq \delta \right\} : \kappa - \delta < k' \right\} \\
&= \sup \left\{ \lim_{\eta \downarrow 0} -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' - (\kappa - \eta)} \vee 0 : \kappa \leq k' \right\},
\end{aligned}$$

using that

$$-\frac{\mathcal{U}^C(k' \vee (\kappa - \eta), \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' \vee (\kappa - \eta) - (\kappa - \eta)} \vee 0 \longrightarrow \lim_{\eta \downarrow 0} -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa - \eta, \tau)}{k' - (\kappa - \eta)} \vee 0.$$

The supremum in the final equality is equivalent to $L^{RTP}(\kappa, \tau)$.

The constructed sequence $\{\mathbb{Q}_{\mathcal{T}}^{u, \eta}\} \subset \mathcal{O}(\mathcal{T})$ thus approaches the lower bound $L^{RTP}(\kappa, \tau)$ in equation (OB.26) as $u \rightarrow \infty$ and $\eta \rightarrow 0$ jointly, and thus also the equivalent expression in equation (3.25). Therefore, these expressions yield the infimum $\mathcal{L}^{RTP}(\kappa, \tau)$ in the right-tail problem (3.21). To determine $\mathcal{L}^{RTP}(\kappa, \tau)$, it now suffices to consider the supremum over $k' \in \mathbb{K} \cup \{\kappa\}$ if $\tau > 0$, and $k' \in \{1\}$ if $\tau = 0$.

OB.4 Proof of Proposition 3.4

For proving the proposition, we closely follow the proof of Proposition 3.3.

Upper bounds: Denote the right-hand side in equation (3.28) by $\bar{U}^{XTP}(\kappa, \tau)$. It immediately follows from the construction of $\bar{\mathbb{Q}}_{\mathcal{T}}$ that $XTP^{\bar{\mathbb{Q}}_{\mathcal{T}}}(\kappa, \tau) \leq \bar{U}^{XTP}(\kappa, \tau)$ for all $\bar{\mathbb{Q}}_{\mathcal{T}} \in \bar{\mathcal{O}}(\mathcal{T})$. Therefore, $\bar{U}^{XTP}(\kappa, \tau)$ is an upper bound over $\bar{\mathcal{O}}(\mathcal{T})$.

Using the construction in the proof of Proposition 3.2, consider the maturity-interpolated lower convex envelope $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^u(\kappa))$ for $\mathbb{L}_{\mathcal{T}}^u(\kappa)$ defined in equation (OB.9). Recall that $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^u(\kappa))$ is associated to some $\bar{\mathbb{Q}}_{\mathcal{T}}^u \in \bar{\mathcal{O}}(\mathcal{T})$ and that $\bar{\mathcal{E}}(\cdot, t; \mathbb{L}_{\mathcal{T}}^u(\kappa)) = \mathcal{E}(\cdot, t; \mathbb{L}_{\mathcal{T}}^u(\kappa))$ for all $t \in \mathbb{T}$.

For left-tail probabilities, analogous to equation (OB.18), we find that

$$\begin{aligned} LTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^u}(\kappa, \tau) &= 1 + \partial_{k+} \left(\frac{t'' - \tau}{t'' - t'} \mathcal{E}(\kappa, t'; \mathbb{L}_{\mathcal{T}}^u(\kappa)) + \frac{\tau - t'}{t'' - t'} \mathcal{E}(\kappa, t''; \mathbb{L}_{\mathcal{T}}^u(\kappa)) \right) \\ &= \frac{t'' - \tau}{t'' - t'} LTP^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, t') + \frac{\tau - t'}{t'' - t'} LTP^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, t''). \end{aligned} \quad (\text{OB.28})$$

For right-tail probabilities, analogous to equation (OB.24), we similarly obtain

$$\begin{aligned} RTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^u}(\kappa, \tau) &= -\partial_{k-} \left(\frac{t'' - \tau}{t'' - t'} \mathcal{E}(\kappa, t'; \mathbb{L}_{\mathcal{T}}^u(\kappa)) + \frac{\tau - t'}{t'' - t'} \mathcal{E}(\kappa, t''; \mathbb{L}_{\mathcal{T}}^u(\kappa)) \right) \\ &= \frac{t'' - \tau}{t'' - t'} RTP^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, t') + \frac{\tau - t'}{t'' - t'} RTP^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, t''). \end{aligned} \quad (\text{OB.29})$$

By the proof of Proposition 3.3, we in both cases have that $XTP^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, t') \rightarrow U^{XTP}(\kappa, t') = \mathcal{U}^{XTP}(\kappa, t')$ and $XTP^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, t'') \rightarrow U^{XTP}(\kappa, t'') = \mathcal{U}^{XTP}(\kappa, t'')$ as $u \rightarrow \infty$, using the definitions of $U^{LTP}(\kappa, t)$ and $U^{RTP}(\kappa, t)$ in equations (OB.17) and (OB.23), respectively. Hence, it follows that $XTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^u}(\kappa, \tau) \rightarrow \bar{U}^{XTP}(\kappa, \tau)$ as $u \rightarrow \infty$. The sequence $\{\bar{\mathbb{Q}}_{\mathcal{T}}^u\} \subset \bar{\mathcal{O}}(\mathcal{T})$ thus approaches the upper bound $\bar{U}^{XTP}(\kappa, \tau)$, which must therefore equal the supremum $\bar{U}^{XTP}(\kappa, \tau)$ in problems (3.21) over $\bar{\mathcal{O}}(\mathcal{T})$.

Lower bounds: Denote the right-hand side in equation (3.29) by $\bar{L}^{XTP}(\kappa, \tau)$. By construction, we have that $XTP^{\bar{\mathbb{Q}}_{\mathcal{T}}}(\kappa, \tau) \geq \bar{L}^{XTP}(\kappa, \tau)$ for all $\bar{\mathbb{Q}}_{\mathcal{T}} \in \bar{\mathcal{O}}(\mathcal{T})$, so that $\bar{L}^{XTP}(\kappa, \tau)$ is a lower bound over $\bar{\mathcal{O}}(\mathcal{T})$.

Now consider the maturity-interpolated lower convex envelopes $\bar{\mathcal{E}}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^u(\kappa \pm \eta))$ with $\mathbb{L}_{\mathcal{T}}^u(\kappa \pm \eta)$ as in equation (OB.9), which are associated to $\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta} \in \bar{\mathcal{O}}(\mathcal{T})$ and satisfy $\bar{\mathcal{E}}(\cdot, t; \mathbb{L}_{\mathcal{T}}^u(\kappa \pm \eta)) = \mathcal{E}(\cdot, t; \mathbb{L}_{\mathcal{T}}^u(\kappa \pm \eta))$ for all $t \in \mathbb{T}$.

For left-tail probabilities, analogous to equation (OB.21), it holds that

$$\begin{aligned} LTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) &= 1 + \partial_{k+} \left(\frac{t'' - \tau}{t'' - t'} \mathcal{E}(\kappa, t'; \mathbb{L}_{\mathcal{T}}^u(\kappa + \eta)) + \frac{\tau - t'}{t'' - t'} \mathcal{E}(\kappa, t''; \mathbb{L}_{\mathcal{T}}^u(\kappa + \eta)) \right) \\ &= \frac{t'' - \tau}{t'' - t'} LTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, t') + \frac{\tau - t'}{t'' - t'} LTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, t''). \end{aligned} \quad (\text{OB.30})$$

For right-tail probabilities, analogous to equation (OB.27), we have that

$$\begin{aligned} RTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) &= -\partial_{k-} \left(\frac{t'' - \tau}{t'' - t'} \mathcal{E}(\kappa, t'; \mathbb{L}_{\mathcal{T}}^u(\kappa - \eta)) + \frac{\tau - t'}{t'' - t'} \mathcal{E}(\kappa, t''; \mathbb{L}_{\mathcal{T}}^u(\kappa - \eta)) \right) \\ &= \frac{t'' - \tau}{t'' - t'} RTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, t') + \frac{\tau - t'}{t'' - t'} RTP^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, t''). \end{aligned} \quad (\text{OB.31})$$

The proof of Proposition 3.3 yields in both cases that $XTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta}}(\kappa, t') \rightarrow L^{XTP}(\kappa, t') = \mathcal{L}^{XTP}(\kappa, t')$ and $XTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta}}(\kappa, t'') \rightarrow L^{XTP}(\kappa, t'') = \mathcal{L}^{XTP}(\kappa, t'')$ as $u \rightarrow \infty$ and $\eta \rightarrow 0$ jointly, using the definitions of $L^{LTP}(\kappa, t)$ and $L^{RTP}(\kappa, t)$ in equations (OB.20) and (OB.26), respectively. As a consequence, we further have that $XTP^{\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) \rightarrow \bar{L}^{XTP}(\kappa, \tau)$ as $u \rightarrow \infty$ and $\eta \rightarrow 0$. Hence, each sequence $\{\bar{\mathbb{Q}}_{\mathcal{T}}^{u, \eta}\} \subset \bar{\mathcal{O}}(\mathcal{T})$

approaches the respective lower bound $\bar{L}^{XTP}(\kappa, \tau)$, which must therefore equal the infimum $\bar{\mathcal{L}}^{XTP}(\kappa, \tau)$ in problems (3.21) over $\bar{\mathcal{O}}(\mathcal{T})$.

OC Further results on option price bounds

This appendix provides further theoretical results regarding the option price bounds in section 3.2.

OC.1 Feasibility conditions

Quoted bid and ask prices may not be (statically) arbitrage-free in the sense that no admissible risk-neutral probability measure exists, i.e., $\mathcal{O}(\mathcal{T}) = \emptyset$. Using the construction of lower convex envelopes according to equation (OA.5) together with the implications of Lemma OA.3, we may establish a tractable criterion that is necessary and sufficient for feasibility of the imposed bid-ask constraints, i.e., $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Effectively, one needs to specify some function $G(k, t) \geq (1 - k)^+$ that decays to zero as $k \rightarrow \infty$ to match the required tail behavior of option prices, but slow enough to stay consistent with the bid-ask constraints in $\mathcal{O}(\mathcal{T})$.

Lemma OC.1. *Let $\mathbb{K} \subset \Omega$ and $\mathbb{T} \subset \mathcal{T}$ be any sets. For a given $u \geq 0$ and some function $G(k, t) \geq (1 - k)^+$ such that $\lim_{k \rightarrow \infty} G(k, t) = 0$ for all $t \in \mathcal{T}$, define*

$$\tilde{\mathbb{A}}_G(u) := \{(k', t', A^C(k', t')) : (k', t') \in \mathbb{K} \times \mathbb{T}\} \cup \{(u', t, G(u', t)) : u' \geq u, t \in \mathcal{T}\}.$$

Then the following are equivalent:

- (i) $\mathcal{O}(\mathcal{T}) \neq \emptyset$;
- (ii) $A^C(k, t) \geq (1 - k)^+$ and $B^C(k, t) \leq \mathcal{E}(k, t; \tilde{\mathbb{A}}_G(u))$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ and some G and u .

Proof. To show that (i) implies (ii), take a particular $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}(\mathcal{T})$. Set $G(k, t) := C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)$ for all $k \geq u = 0$ and $t \in \mathcal{T}$. Thereby, we automatically have that $G(k, t) \geq (1 - k)^+$ and $\lim_{k \rightarrow \infty} G(k, t) = 0$ for all $t \in \mathcal{T}$. Moreover, due to the resulting identity $\mathcal{E}(k, t; \tilde{\mathbb{A}}_G(u)) = C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)$ for all $(k, t) \in \Omega \times \mathcal{T}$, it holds that $(1 - k)^+ \vee B^C(k, t) \leq \mathcal{E}(k, t; \tilde{\mathbb{A}}_G(u)) \leq A^C(k, t)$.

To show that (ii) implies (i), we may use Lemma OA.3. The assumptions $G(k, t) \geq (1 - k)^+$ and $\lim_{k \rightarrow \infty} G(k, t) = 0$ for all $t \in \mathcal{T}$ together with $A(k, t) \geq (1 - k)^+$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ thus yield that $\mathcal{E}(\cdot, \cdot; \tilde{\mathbb{A}}_G(u))$ is an arbitrage-free option price surface in the sense of Lemma OA.2, associated to a probability measure $\tilde{\mathbb{Q}}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$. In addition, we have that $B(k, t) \leq C^{\tilde{\mathbb{Q}}_{\mathcal{T}}}(k, t) \leq A(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, where the first inequality holds by hypothesis and the second inequality by construction of the lower convex envelope as $\mathcal{E}(k, t; \tilde{\mathbb{A}}_G(u)) \leq A(k, t)$. Therefore, we conclude that $\tilde{\mathbb{Q}}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$ and thus $\mathcal{O}(\mathcal{T}) \neq \emptyset$. $\square$

When $\mathbb{K}$ is bounded and $\mathbb{T}$ is finite, an even simpler condition may be formulated. In essence, we may choose in Lemma OC.1 $G(k, t) = 0$ for some large enough u . Concretely, if $\mathcal{O}(\mathcal{T}) \neq \emptyset$, then any $u \geq \max\{z^*(t) : t \in \mathbb{T}\}$ with $z^*(t)$ determined according to Lemma OC.5 is a valid choice.

Lemma OC.2. *Let $\mathbb{K} \subset [0, \bar{K}]$ be bounded and $\mathbb{T} \subset \mathcal{T}$ finite. For a given $u \geq 1$, define*

$$\tilde{\mathbb{A}}_0(u) := \{(k', t', A^C(k', t')) : (k', t') \in \mathbb{K} \times \mathbb{T}\} \cup \{(u, t, 0) : t \in \mathcal{T}\}.$$

Then the following are equivalent:

(i) $\mathcal{O}(\mathcal{T}) \neq \emptyset$;

(ii) $A^C(k, t) \geq (1 - k)^+$ and $B^C(k, t) \leq \mathcal{E}(k, t; \tilde{A}_0(u))$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ and some large enough u .

Proof. To show that (i) implies (ii), consider a particular $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}(\mathcal{T})$ and set $k^* := \sup \mathbb{K} \leq \bar{K}$. To obtain a threshold u^* for u , take the maximum over all convex extrapolations of option prices $C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)$ at strike k^* and maturities $t \in \mathbb{T} \setminus \{0\}$, from which we get

$$u^* := k^* \vee \max \left\{ k^* - \frac{C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t) - (1 - k^*)^+}{\Delta_{k^* - C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t)}} : t \in \mathbb{T} \setminus \{0\} \right\} \geq k^*,$$

where we define $\Delta_{k^* - C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t) := \sup \left\{ \frac{C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t) - C^{\mathbb{Q}_{\mathcal{T}}^*}(k', t)}{k^* - k'} : k' < k^* \right\}$ over $k' \in \mathbb{K} \cup \{0\}$. If a denominator in the expression for u^* is zero, so must be the numerator and we use the convention that the fraction is zero.

For any $u \geq u^*$, determine the lower convex envelope $\mathcal{E}(\cdot, \cdot; \tilde{A}_0(u))$ as well as another lower convex envelope $\mathcal{E}(\cdot, \cdot; \tilde{C}(u))$ when setting

$$\tilde{C}(u) := \{(k', t', C^{\mathbb{Q}_{\mathcal{T}}^*}(k', t')) : (k', t') \in \mathbb{K} \times \mathbb{T}\} \cup \{(u, t, 0) : t \in \mathcal{T}\},$$

for which we immediately obtain that $\mathcal{E}(k, t; \tilde{C}(u)) = C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. By the preceding construction, it follows from the monotonicity of lower convex envelopes with $C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t) \leq A^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ that $C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t) \leq \mathcal{E}(k, t; \tilde{A}_0(u))$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. The construction of lower convex envelopes further yields that $\mathcal{E}(k, t; \tilde{A}_0(u)) \leq A^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. Therefore, due to the fact that $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}(\mathcal{T})$, we specifically have that $(1 - k)^+ \vee B^C(k, t) \leq \mathcal{E}(k, t; \tilde{A}_0(u)) \leq A^C(k, t)$.

To show that (ii) implies (i), we invoke again Lemma OA.3. Thereby, using that by assumption $A(k, t) \geq (1 - k)^+$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, we have that $\mathcal{E}(\cdot, \cdot; \tilde{A}_0(u))$ constitutes an arbitrage-free option price surface, associated to a probability measure $\tilde{\mathbb{Q}}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$. It holds that $B(k, t) \leq C^{\tilde{\mathbb{Q}}_{\mathcal{T}}}(k, t) \leq A(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, where the first inequality is satisfied by hypothesis and the second inequality by construction of the lower convex envelope as $\mathcal{E}(k, t; \tilde{A}_0(u)) \leq A(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. Hence, $\tilde{\mathbb{Q}}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$ follows, so that $\mathcal{O}(\mathcal{T}) \neq \emptyset$. $\square$

OC.2 Some properties of option price bounds

The lower and upper option price bounds in Proposition 3.1 exhibit substantial structural differences. These differences are reflected in the fact that upper option price bounds may be uniformly approximated over any bounded strike interval $[0, K]$ by option prices under some $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$, as formalized in the following lemma. An analogous approximation result does generally not hold for lower option price bounds.

Lemma OC.3. *Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Take a bounded set $[0, K]$ for arbitrary $K > 0$. Then for each $\varepsilon > 0$, there exists a $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$ such that $\mathcal{U}^{\chi}(k, t) - \chi^{\mathbb{Q}_{\mathcal{T}}}(k, t) \leq \varepsilon$ uniformly for all $(k, t, \chi) \in [0, K] \times \mathcal{T} \times \{C, P\}$.*

Proof. It suffices to verify that $\mathcal{U}^C(k, t) - C^{\mathbb{Q}_{\mathcal{T}}}(k, t) \leq \varepsilon$ uniformly on $[0, K] \times \mathcal{T}$ for the call price bounds in equation (3.6). The analogous statement that $\mathcal{U}^P(k, t) - P^{\mathbb{Q}_{\mathcal{T}}}(k, t) \leq \varepsilon$ uniformly on $[0, K] \times \mathcal{T}$ for the put price bounds in equation (3.8) then follows immediately from the put-call parity relations (3.5).

Take $u > K$ and define $U^C(k, t; u)$ for $(k, t) \in \Omega \times \mathcal{T}$ as in equation (OB.3) for an arbitrary $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}(\mathcal{T})$.

Equivalently to equation (OB.3), we may write for $k' \in \mathbb{K} \cup \{0, 1\}$ and $u' \geq u$

$$U^C(k, t; u) = \inf \left\{ \frac{u' - k}{u' - k'} \mathcal{U}^C(k', t) + \frac{k - k'}{u' - k'} C^{\mathbb{Q}_T^*}(u', t') : k' \leq k < u', t' \geq t \right\}. \quad (\text{OC.1})$$

Therefore, we obtain from equation (OC.1) for the error $\mathcal{U}^C(k, t) - U^C(k, t; u) \geq 0$ at $(k, t) \in [0, K] \times \mathcal{T}$ that

$$\begin{aligned} \mathcal{U}^C(k, t) - U^C(k, t; u) &= \sup \left\{ \mathcal{U}^C(k, t) - \frac{u' - k}{u' - k'} \mathcal{U}^C(k', t) - \frac{k - k'}{u' - k'} C^{\mathbb{Q}_T^*}(u', t') \right\} \\ &\leq \sup \left\{ \frac{k - k'}{u' - k'} \mathcal{U}^C(k', t) - \frac{k - k'}{u' - k'} C^{\mathbb{Q}_T^*}(u', t') \right\} \\ &\leq \frac{K}{u}, \end{aligned} \quad (\text{OC.2})$$

independent of (k, t) . In particular, the first inequality in equation (OC.2) results as $\mathcal{U}^C(k, t) \leq \mathcal{U}^C(k', t)$ for $k \geq k'$. The second inequality is due to the fact that $C^{\mathbb{Q}_T^*}(u', t') \geq 0$ and $\mathcal{U}^C(k', t) \leq 1$.

As argued in the proof of Proposition 3.1, $U^C(\cdot, \cdot; u)$ is associated to an admissible $\mathbb{Q}_T^u \in \mathcal{O}(\mathcal{T})$ such that $C^{\mathbb{Q}_T^u}(k, t) = U^C(k, t; u)$ for all $(k, t) \in \Omega \times \mathcal{T}$. Hence, using equation (OC.2), any $u \geq K/\varepsilon$ for given $\varepsilon > 0$ yields $\mathbb{Q}_T^u \in \mathcal{O}(\mathcal{T})$ satisfying $\mathcal{U}^C(k, t) - C^{\mathbb{Q}_T^u}(k, t) \leq \varepsilon$ uniformly for all $(k, t) \in [0, K] \times \mathcal{T}$. $\square$

An even stronger approximation result than in Lemma OC.3 holds. In fact, as established in the following lemma, the upper option price bounds in Proposition 3.1 are exactly matched by option prices under some $\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})$ over any bounded strike interval $[0, K_\varepsilon]$ on which upper call price bounds are strictly decreasing with a slope that is uniformly bounded away from zero by $-\varepsilon < 0$. When $\mathbb{T}$ is a finite set, the conclusion also holds with $\varepsilon = 0$.

Lemma OC.4. *Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Take the bounded set $[0, K_\varepsilon]$ for $K_\varepsilon := \sup \{k \in \mathbb{K} : \partial_{k-} \mathcal{U}^C(k, t) < -\varepsilon, t \in \mathbb{T}\} \geq 0$ given some small $0 < \varepsilon < 1$. Then there exists a $\mathbb{Q}_T \in \mathcal{O}(\mathcal{T})$ such that $\mathcal{U}^\chi(k, t) = \chi^{\mathbb{Q}_T}(k, t)$ for all $(k, t, \chi) \in [0, K_\varepsilon] \times \mathcal{T} \times \{C, P\}$.*

Proof. It again suffices to verify that $\mathcal{U}^C(k, t) = C^{\mathbb{Q}_T}(k, t)$ on $[0, K_\varepsilon] \times \mathcal{T}$ for the call price bounds in equation (3.6). The analogous statement that $\mathcal{U}^P(k, t) = P^{\mathbb{Q}_T}(k, t)$ on $[0, K_\varepsilon] \times \mathcal{T}$ for the put price bounds in equation (3.8) is then a direct consequence of the put-call parity relations (3.5).

To establish the result, we use $U^C(k, t; u)$ for $(k, t) \in \Omega \times \mathcal{T}$ according to equation (OB.3) for an arbitrary $\mathbb{Q}_T^* \in \mathcal{O}(\mathcal{T})$, where $u \geq u^*$ is chosen large enough. Here, the threshold u^* is determined as

$$u^* := K_\varepsilon \vee \sup \left\{ K_\varepsilon - \frac{\mathcal{U}^C(K_\varepsilon, t) - (1 - K_\varepsilon)^+}{\partial_{k-} \mathcal{U}^C(K_\varepsilon, t)} : t \in \mathbb{T} \setminus \{0\} \right\} \geq K_\varepsilon,$$

noting that its finiteness is guaranteed by assumption. Due to this construction, we have that $U^C(k, t; u) = \mathcal{U}^C(k, t)$ for all $(k, t) \in [0, K_\varepsilon] \times \mathcal{T}$.

Finally, as in the proof of Proposition 3.1, we may use that $U^C(\cdot, \cdot; u)$ is associated to an admissible $\mathbb{Q}_T^u \in \mathcal{O}(\mathcal{T})$ satisfying $C^{\mathbb{Q}_T^u}(k, t) = U^C(k, t; u)$ for all $(k, t) \in \Omega \times \mathcal{T}$. Therefore, we immediately obtain the desired equality $\mathcal{U}^C(k, t) = C^{\mathbb{Q}_T^u}(k, t)$ for all $(k, t) \in [0, K_\varepsilon] \times \mathcal{T}$ whenever $u \geq u^*$. $\square$

OC.3 Tail behavior of option price bounds

Throughout the following discussion, consider a strike range $\mathbb{K} \subset [k_*, k^*] := [\inf \mathbb{K}, \sup \mathbb{K}] \subset (0, \bar{K})$, bounded away from the respective boundaries in both tails, and a finite maturity set $\mathbb{T}$. In that case, the

option price bounds in Proposition 3.1 exhibit notable tail behavior. In particular, the explicit expressions for lower and upper bounds yield the following immediate corollary to Proposition 3.1, which characterizes the left-tail and right-tail behavior of the option price bounds.

Corollary OC.1. *Let $\mathbb{K} \subset [k_*, k^*] := [\inf \mathbb{K}, \sup \mathbb{K}] \subset (0, \bar{K})$ be bounded and $\mathbb{T} \subset \mathcal{T}$ finite. Moreover, let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.4):*

(i) *For any small strike $\kappa \leq k_*$ and $\tau > 0$, option price bounds are given by*

$$\mathcal{U}^C(\kappa, \tau) = \mathcal{U}^C(k_*, \tau) + \frac{\mathcal{U}^C(k_*, \tau) - 1}{k_*} (\kappa - k_*), \quad (\text{OC.3})$$

$$\mathcal{L}^C(\kappa, \tau) = (1 - \kappa)^+ \vee \max\{\mathcal{L}^C(k_*, t') + D_{k+}\mathcal{L}^C(k_*, t') (\kappa - k_*) : 0 < t' \leq \tau\}, \quad (\text{OC.4})$$

$$\text{where } D_{k+}\mathcal{L}^C(k_*, \tau) := 0 \wedge \inf\left\{\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(k_*, \tau)}{k' - k_*} : k_* < k'\right\}.$$

(ii) *For any large strike $\kappa \geq k^*$ and $\tau > 0$, option price bounds are given by*

$$\mathcal{U}^C(\kappa, \tau) = \mathcal{U}^C(k^*, \tau), \quad (\text{OC.5})$$

$$\mathcal{L}^C(\kappa, \tau) = 0 \vee \max\{\mathcal{L}^C(k^*, t') + D_{k-}\mathcal{L}^C(k^*, t') (\kappa - k^*) : 0 < t' \leq \tau\}, \quad (\text{OC.6})$$

$$\text{where } D_{k-}\mathcal{L}^C(k^*, \tau) := \left\{\frac{\mathcal{L}^C(k^*, \tau) - 1}{k^*}\right\} \vee \sup\left\{\frac{\mathcal{L}^C(k^*, \tau) - \mathcal{U}^C(k', \tau)}{k^* - k'} : k' < k^*\right\}.$$

From Corollary OC.1, observe that $\mathcal{U}^C(\kappa, \tau)$ in equation (OC.3) corresponds to a linear interpolation between $\mathcal{U}^C(0, \tau) = 1$ and $\mathcal{U}^C(k_*, \tau)$ for any $0 \leq \kappa \leq k_*$ in the left tail. Moreover, note that $\mathcal{U}^C(\kappa, \tau)$ in equation (OC.5) is constant at $\mathcal{U}^C(k^*, \tau)$ for all $\kappa \geq k^*$. In contrast, $\mathcal{L}^C(\kappa, \tau)$ in equations (OC.4) and (OC.6) may exhibit somewhat more complex behavior in both tails, while eventually becoming trivial, i.e., equal to $(1 - \kappa)^+$, for all $\kappa \leq z_*(\tau)$ in the left tail and $\kappa \geq z^*(\tau)$ in the right tail. We have the following explicit characterization of $z_*(\tau)$ and $z^*(\tau)$.

Lemma OC.5. *Let $\mathbb{K} \subset [k_*, k^*] := [\inf \mathbb{K}, \sup \mathbb{K}] \subset (0, \bar{K})$ be bounded and $\mathbb{T} \subset \mathcal{T}$ finite. Moreover, let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.4):*

(i) *$z_*(\tau) := 1 \wedge \inf\{k \in \Omega : \mathcal{L}^C(k, \tau) > (1 - k)^+\}$ with $\tau \geq 0$ is given by*

$$z_*(\tau) = 1 \wedge \min\left\{\tilde{z}_*(t') - \frac{\mathcal{L}^C(\tilde{z}_*(t'), t') + \tilde{z}_*(t') - 1}{D_{k+}\mathcal{L}^C(\tilde{z}_*(t'), t') + 1} : 0 < t' \leq \tau\right\}, \quad (\text{OC.7})$$

$$\text{where } \tilde{z}_*(\tau) := 1 \wedge \inf\{k \in \mathbb{K} : \mathcal{L}^C(k, \tau) > (1 - k)^+\}.$$

(ii) *$z^*(\tau) := 1 \vee \sup\{k \in \Omega : \mathcal{L}^C(k, \tau) > (1 - k)^+\}$ with $\tau \geq 0$ is given by*

$$z^*(\tau) = 1 \vee \max\left\{\tilde{z}^*(t') - \frac{\mathcal{L}^C(\tilde{z}^*(t'), t')}{D_{k-}\mathcal{L}^C(\tilde{z}^*(t'), t')} : 0 < t' \leq \tau\right\}, \quad (\text{OC.8})$$

$$\text{where } \tilde{z}^*(\tau) := 1 \vee \sup\{k \in \mathbb{K} : \mathcal{L}^C(k, \tau) > (1 - k)^+\}.$$

Extending the definition of $\mathcal{O}(\mathcal{T})$ in equation (3.3), for some $0 \leq \ell \leq 1 \leq u \leq \infty$, define the set of

admissible risk-neutral probability measures having bounded support contained in the interval $[\ell, u]$ by

$$\mathcal{O}(\mathcal{T}; [\ell, u]) := \mathcal{O}(\mathcal{T}) \cap \{\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T}) : C^{\mathbb{Q}_{\mathcal{T}}}(\ell, t) = 1 - \ell, C^{\mathbb{Q}_{\mathcal{T}}}(u, t) = 0, t \in \mathcal{T}\}. \quad (\text{OC.9})$$

As a special case, we have $\mathcal{O}(\mathcal{T}) = \mathcal{O}(\mathcal{T}; [0, \infty))$. Moreover, observe that $\mathcal{O}(\mathcal{T}; [\ell, u])$ is increasing as $\ell \rightarrow 0$ or $u \rightarrow \infty$. Clearly, in order for the feasibility condition $\mathcal{O}(\mathcal{T}; [\ell, u]) \neq \emptyset$ to hold, we need that $\ell \leq z_*(t)$ and $u \geq z^*(t)$ for all $t \in \mathbb{T}$ as in Lemma OC.5. While $\mathcal{O}(\mathcal{T}; [\min\{z_*(t) : t \in \mathbb{T}\}, \max\{z^*(t) : t \in \mathbb{T}\}]) \neq \emptyset$ may generally not hold, the following lemma yields that some probability measures with bounded support are always feasible provided that $\mathcal{O}(\mathcal{T}) \neq \emptyset$. In particular, due to the construction of lower option price bounds in the proof of Proposition 3.1, $\mathcal{O}(\mathcal{T}) \neq \emptyset$ implies that $\mathcal{O}(\mathcal{T}; [0, \max\{z^*(t) : t \in \mathbb{T}\}]) \neq \emptyset$, which already follows from Lemma OC.2.

Lemma OC.6. *Let $\mathbb{K} \subset [k_*, k^*] := [\inf \mathbb{K}, \sup \mathbb{K}] \subset (0, \bar{K})$ be bounded and $\mathbb{T} \subset \mathcal{T}$ finite. Moreover, suppose that $\mathcal{O}(\mathcal{T}; [\ell, \infty)) \neq \emptyset$ for some $0 \leq \ell \leq 1$. Then there exists some $u \geq 1$ such that $\mathcal{O}(\mathcal{T}; [\ell, u]) \neq \emptyset$.*

Proof. Take any $\bar{T} \in \mathcal{T}$ such that $\bar{T} \geq \max \mathbb{T}$. By hypothesis, there exists some $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}(\mathcal{T}; [\ell, \infty))$. Define $\mathbb{Q}_{\mathcal{T}}$ by modifying the tails of $\mathbb{Q}_{\mathcal{T}}^*$ such that, for $t' := \bar{T} \wedge \min\{t'' \in \mathbb{T} \cup \{0\} : t'' \geq t\}$,

$$P^{\mathbb{Q}_{\mathcal{T}}}(k, t) := \begin{cases} P^{\mathbb{Q}_{\mathcal{T}}^*}(k, t') & \text{if } k_* \leq k \leq k^* \\ P^{\mathbb{Q}_{\mathcal{T}}^*}(k_*, t') + b_L(t')(k - k_*) & \text{if } \ell < k < k_* \\ 0 & \text{if } k \leq \ell \end{cases}, \quad (\text{OC.10})$$

$$C^{\mathbb{Q}_{\mathcal{T}}}(k, t) := \begin{cases} C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t') & \text{if } k_* \leq k \leq k^* \\ C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t') + b_R(t')(k - k^*) & \text{if } k^* < k < u \\ 0 & \text{if } k \geq u \end{cases}, \quad (\text{OC.11})$$

where $b_L(t) := \frac{P^{\mathbb{Q}_{\mathcal{T}}^*}(k_*, t)}{k_* - \ell}$ and $b_R(t) := -\frac{C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t)}{u - k^*}$ for $u \geq u^*$ with

$$u^* := k^* \vee \max \left\{ k^* - \frac{C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t) - (1 - k^*)^+}{\partial_{k-} C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t)} : t \in \mathbb{T} \setminus \{0\} \right\} \geq k^*,$$

setting the respective fraction to zero if $\partial_{k-} C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t) = 0$. Essentially, for $(k, t) \in \Omega \times \mathcal{T}$, $C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ thus corresponds to the lower convex envelope $\mathcal{E}(k, t \wedge \bar{T}; \tilde{\mathbb{C}}(\ell, u))$ with

$$\tilde{\mathbb{C}}(\ell, u) := \{(k, t, C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)) : (k, t) \in [k_*, k^*] \times (\mathbb{T} \cup \bar{T})\} \cup \{(\ell, t, 1 - \ell), (u, t, 0) : t \in \mathcal{T}\}.$$

By Lemma OA.3, this construction thus ensures that $C^{\mathbb{Q}_{\mathcal{T}}}$ satisfies the no-arbitrage conditions of Lemma OA.2. Hence, it holds that $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$, setting $\mathbb{Q}_t(R_t \leq k) = 1 + \partial_{k+} C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ for all $(k, t) \in \Omega \times \mathcal{T}$ as in equation (OA.4). Moreover, $C^{\mathbb{Q}_{\mathcal{T}}}(k, t) = C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)$ for all $(k, t) \in [k_*, k^*] \times \mathbb{T} \supset \mathbb{K} \times \mathbb{T}$ together with $C^{\mathbb{Q}_{\mathcal{T}}}(\ell, t) = 1 - \ell$ and $C^{\mathbb{Q}_{\mathcal{T}}}(u, t) = 0$ for all $t \in \mathcal{T}$ further implies that $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T}; [\ell, u])$, as required. $\square$

Using $\mathcal{O}(\mathcal{T}; [\ell, u])$ as in equation (OC.9), we may determine option price bounds analogous to problems (3.4) across all admissible probability measures having support contained in $[\ell, u]$. Formally, these bounds are obtained from the problems

$$\inf_{\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T}; [\ell, u])} C^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau), \quad \inf_{\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T}; [\ell, u])} P^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau). \quad (\text{OC.12})$$

The resulting lower bound is denoted by $\mathcal{L}^\chi(\kappa, \tau; [\ell, u])$ and the upper bound by $\mathcal{U}^\chi(\kappa, \tau; [\ell, u])$, noting the relations $\mathcal{L}^\chi(\kappa, \tau) = \mathcal{L}^\chi(\kappa, \tau; [0, \infty))$ and $\mathcal{U}^\chi(\kappa, \tau) = \mathcal{U}^\chi(\kappa, \tau; [0, \infty))$.

Remarkably, even though $\lim_{u \rightarrow \infty} \mathcal{O}(\mathcal{T}; [0, u]) \subsetneq \mathcal{O}(\mathcal{T})$ since the limit does not contain measures with unbounded support, the option price bounds in Proposition 3.1 arise as a limit of $\mathcal{L}^\chi(\kappa, \tau; [0, u])$ and $\mathcal{U}^\chi(\kappa, \tau; [0, u])$ as $u \rightarrow \infty$. The following corollary formalizes this in a generalized form, where it should be recalled that the condition $\mathcal{O}(\mathcal{T}; [\ell^*, u^*]) \neq \emptyset$ is implied by the condition $\mathcal{O}(\mathcal{T}; [\ell^*, \infty)) \neq \emptyset$, due to Lemma OC.6. The corollary also generalizes to the case where $\mathbb{T}$ is not finite.

Corollary OC.2. *Let $\mathbb{K} \subset [k_*, k^*] := [\inf \mathbb{K}, \sup \mathbb{K}] \subset (0, \bar{K})$ be bounded and $\mathbb{T} \subset \mathcal{T}$ finite. Moreover, let $\mathcal{O}(\mathcal{T}; [\ell^*, u^*]) \neq \emptyset$ for some $0 \leq \ell^* \leq 1 \leq u^* < \infty$. Then, for any $0 \leq \ell \leq \ell^*$ and $u \geq u^*$, the option price bounds in problems (OC.12) with $\kappa \geq 0$ and $\tau \geq 0$ are given by*

$$\begin{aligned} \mathcal{U}^C(\kappa, \tau; [\ell, u]) &= \mathcal{U}^C(\kappa, \tau) \wedge \max \left\{ (1 - \kappa)^+, \frac{u - \kappa}{u - \ell} (1 - \ell) \right\} \\ &\quad \wedge \inf \left\{ \frac{k' - \kappa}{k' - \ell} (1 - \ell) + \frac{\kappa}{k'} A^C(k', t') : \ell < \kappa \leq k', t' \geq \tau \right\} \\ &\quad \wedge \inf \left\{ \frac{u - \kappa}{u - k'} A^C(k', t') : k' \leq \kappa < u, t' \geq \tau \right\}, \end{aligned} \quad (\text{OC.13})$$

$$\begin{aligned} \mathcal{L}^C(\kappa, \tau; [\ell, u]) &= \mathcal{L}^C(\kappa, \tau) \vee \sup \left\{ \frac{k' - \kappa}{k' - \ell} (1 - \ell) + \frac{\kappa - \ell}{k' - \ell} B^C(k', t') : \ell < k' \leq \kappa, t' \leq \tau \right\} \\ &\quad \vee \sup \left\{ \frac{u - \kappa}{u - k'} B^C(k', t') : \kappa \leq k' < u, t' \leq \tau \right\}. \end{aligned} \quad (\text{OC.14})$$

Moreover, it holds that $\mathcal{U}^C(\kappa, \tau; [\ell, u]) \rightarrow \mathcal{U}^C(\kappa, \tau)$ and $\mathcal{L}^C(\kappa, \tau; [\ell, u]) \rightarrow \mathcal{L}^C(\kappa, \tau)$ as $\ell \rightarrow 0$ and $u \rightarrow \infty$ over $0 \leq \ell \leq \ell^*$ and $u \geq u^*$.

For a given $\mathbb{Q}_\tau$, Lee (2004) and recent extensions in Raval and Jacquier (2023) relate the finiteness of certain higher-order moments of R_τ to the tail behavior of the associated option prices under $\mathbb{Q}_\tau$ for small and large strikes. Except for boundary cases where feasibility may be affected, additionally imposing finiteness of moments such as $\mathbb{E}^{\mathbb{Q}_\tau}[(R_\tau)^p]$ or $\mathbb{E}^{\mathbb{Q}_\tau}[(\log R_\tau)^p]$ for an order $p \in \mathbb{R}$ is inconsequential for the moment bounds in Proposition 3.1. Specifically, if $\mathcal{O}(\mathcal{T}; [\ell, \infty)) \neq \emptyset$ for some $\ell > 0$, Corollary OC.2 yields that the construction of these option price bounds may rely on limits of probability measures in $\mathcal{O}(\mathcal{T}; [\ell, u])$ as $\ell \rightarrow 0$ and $u \rightarrow \infty$, for which moments of all arbitrary orders p are finite. To have an effect on option price bounds in this case, a uniform bound for some higher-order moment across all probability measures would need to be imposed (e.g., as in Henrion et al. (2021)). In that case, the obtained option price bounds would be parameterized by the exogenously imposed uniform bound, whose choice is generally arbitrary. Extensions along these lines are beyond the scope of this paper.

Provided that $\mathcal{O}(\mathcal{T}; [\ell, \infty)) \neq \emptyset$ for some $\ell > 0$, the upper bound $\mathcal{U}^C(\kappa, \tau; [\ell, u])$ in Corollary OC.2 induces probability measures $\mathbb{Q}_\tau^{\ell, u} \in \mathcal{O}(\mathcal{T}; [\ell, u]) \subset \mathcal{O}(\mathcal{T})$, parameterized by ℓ and u , such that $\mathbb{E}^{\mathbb{Q}_\tau^{\ell, u}}[(R_\tau)^p]$ is finite but becomes arbitrarily large as $\ell \rightarrow 0$ and $u \rightarrow \infty$. This behavior extends to any function $f(R)$ that is unbounded as $R \rightarrow 0$ or has super-linear growth as $R \rightarrow \infty$ (see Bondarenko et al. (2024)). In an informal way, using the connection of a general class of moments and option prices via the well-known static replication formula (cf. Bick (1982), Britten-Jones and Neuberger (2000), and Carr and Madan (2001b)), these observations suggest that $\mathcal{O}(\mathcal{T})$ may be consistent with a wide range of tail behavior of option prices.

To formalize this idea, we introduce a broad class $\Phi := \Phi_L \times \Phi_R$ of tail densities that prescribe a certain left-tail and right-tail behavior over some non-trivial subintervals $(0, r_*) \subset (0, k_*)$ and $(r^*, \infty) \subset (k^*, \infty)$,

respectively. For each $\phi_L \in \Phi_L$ and $\phi_R \in \Phi_R$, we suppose that $\phi_L(R) \geq 0$ and $\phi_R(R) \geq 0$, while both functions are locally bounded with finite integrals

$$\int_0^{k_*} \phi_L(R) dR, \quad \int_0^{k_*} R \phi_L(R) dR, \quad \int_{k^*}^{\infty} \phi_R(R) dR, \quad \int_{k^*}^{\infty} R \phi_R(R) dR.$$

The tail densities are then associated to well-defined put and call option prices for $k \leq k_*$ and $k \geq k^*$ via

$$\tilde{P}^{\phi_L}(k) := \int_0^k (k - R) \phi_L(R) dR, \quad \tilde{C}^{\phi_R}(k) := \int_k^{\infty} (R - k) \phi_R(R) dR.$$

We restrict the attention to non-trivial tail densities by assuming that we have $\tilde{P}^{\phi_L}(k) > 0$ for all $0 < k \leq k_*$ and $\tilde{C}^{\phi_R}(k) > 0$ for all $k \geq k^*$. Under some weak technical conditions, the following lemma establishes that any $(\phi_L, \phi_R) \in \Phi$ is consistent with $\mathcal{O}(\mathcal{T})$, including the widely employed power-law densities $\phi_L(R) \propto R^\alpha$ with $\alpha > -1$ and $\phi_R(R) \propto R^\beta$ with $\beta < -2$, among many others.

Lemma OC.7. *Let $\mathbb{K} \subset [k_*, k^*] := [\inf \mathbb{K}, \sup \mathbb{K}] \subset (0, \bar{K})$ be bounded and $\mathbb{T} \subset \mathcal{T}$ finite. Moreover, suppose that $\mathcal{O}(\mathcal{T}, [\ell, \infty)) \neq \emptyset$ for some $\ell > 0$ as well as $\mathcal{U}^P(k_*, t) > 0$ and $\mathcal{U}^C(k^*, t) > 0$ for all $t \in \mathbb{T}$. Then, for any $(\phi_L, \phi_R) \in \Phi$, there exists a $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$ such that $\mathbb{Q}_t$ at $t > 0$ has density $c_L(t) \phi_L(R)$ for all $R < r_* < k_*$ and $c_R(t) \phi_R(R)$ for all $R > r^* > k^*$, where $c_L(t), c_R(t) > 0$ are normalizing constants.*

Proof. Take any $\bar{T} \in \mathcal{T}$ with $\bar{T} \geq \max \mathbb{T}$. By hypothesis and Lemma OC.6, there exists a $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}(\mathcal{T}; [\ell, u])$ for some $u > k^*$, which may be chosen such that $P^{\mathbb{Q}_{\mathcal{T}}^*}(k_*, t) > 0$ and $C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t) > 0$ for all $t \in \mathcal{T} \setminus \{0\}$. Construct a modification $\mathbb{Q}_{\mathcal{T}}$, parameterized by (r_*, r^*) with $\ell < r_* < k_*$ and $k^* < r^* < u$, whose tails behave according to $(\phi_L, \phi_R) \in \Phi$. For this, defining $t' := \bar{T} \wedge \min\{t'' \in \mathbb{T} \cup \{0\} : t'' \geq t\}$, we set

$$P^{\mathbb{Q}_{\mathcal{T}}}(k, t) := \begin{cases} P^{\mathbb{Q}_{\mathcal{T}}^*}(k, t') & \text{if } k_* \leq k \leq k^* \\ P^{\mathbb{Q}_{\mathcal{T}}^*}(k_*, t') + b_L(t')(k - k_*) & \text{if } r_* \leq k < k_* \\ P^{\mathbb{Q}_{\mathcal{T}}}(r_*, t) \frac{\tilde{P}^{\phi_L}(k)}{\tilde{P}^{\phi_L}(r_*)} & \text{if } k < r_* \end{cases}, \quad (\text{OC.15})$$

$$C^{\mathbb{Q}_{\mathcal{T}}}(k, t) := \begin{cases} C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t') & \text{if } k_* \leq k \leq k^* \\ C^{\mathbb{Q}_{\mathcal{T}}^*}(k^*, t') + b_R(t')(k - k^*) & \text{if } k^* < k \leq r^* \\ C^{\mathbb{Q}_{\mathcal{T}}}(r^*, t) \frac{\tilde{C}^{\phi_R}(k)}{\tilde{C}^{\phi_R}(r^*)} & \text{if } k > r^* \end{cases}, \quad (\text{OC.16})$$

using $b_L(t)$ and $b_R(t)$ as in the proof of Lemma OC.6. By construction, we have at any $t \in \mathcal{T} \setminus \{0\}$ that $P^{\mathbb{Q}_{\mathcal{T}}}(k, t) > 0$ for all $0 < k < k_*$ and $C^{\mathbb{Q}_{\mathcal{T}}}(k, t) > 0$ for all $k > k^*$. In order for $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$, defined as in equation (OA.4) via $\mathbb{Q}_t(R_t \leq k) = 1 + \partial_{k+} C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ for all $(k, t) \in \Omega \times \mathcal{T}$, it suffices to verify that $P^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ and $C^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ are convex in the strike dimension at $k = r_*$ and $k = r^*$, respectively. All remaining conditions of Lemma OA.2 are immediately satisfied by $C^{\mathbb{Q}_{\mathcal{T}}}$. Since option prices on $\mathbb{K} \times \mathbb{T}$ remain unaffected by the tail modification as $C^{\mathbb{Q}_{\mathcal{T}}}(k, t) = C^{\mathbb{Q}_{\mathcal{T}}^*}(k, t)$ for all $(k, t) \in [k_*, k^*] \times \mathbb{T} \supset \mathbb{K} \times \mathbb{T}$, we have that $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$ implies also $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})$.

For the convexity of $P^{\mathbb{Q}\tau}(k, t)$ at $k = r_*$, it needs to hold that

$$\partial_{k-} P^{\mathbb{Q}\tau}(r_*, t) = P^{\mathbb{Q}\tau}(r_*, t) \frac{\partial_{k-} \tilde{P}^{\phi_L}(r_*)}{\tilde{P}^{\phi_L}(r_*)} \leq \partial_{k+} P^{\mathbb{Q}\tau}(r_*, t) = b_L(t). \quad (\text{OC.17})$$

Rewriting equation (OC.17) yields the equivalent condition

$$r_* - \frac{\tilde{P}^{\phi_L}(r_*)}{\partial_{k-} \tilde{P}^{\phi_L}(r_*)} = \frac{\int_0^{r_*} R \phi_L(R) dR}{\int_0^{r_*} \phi_L(R) dR} \leq r_* - \frac{P^{\mathbb{Q}\tau}(r_*, t)}{b_L(t)} = \ell. \quad (\text{OC.18})$$

Note that the left-hand side of equation (OC.18) is monotonously decreasing as $r_* \downarrow \ell$. Since the expression is independent of t , it suffices to verify that equation (OC.18) holds at some admissible r_* . In order to prove the claim by contradiction, suppose that equation (OC.18) does not hold for all $\ell < r_* < k_*$, so that

$$\int_0^{r_*} R \phi_L(R) dR > \ell \int_0^{r_*} \phi_L(R) dR. \quad (\text{OC.19})$$

Letting $r_* \downarrow \ell$, equation (OC.19) continues to hold as a weak inequality at $r_* = \ell$, due to continuity in r_* . Since $R \leq \ell$ over the relevant integration domain, in fact

$$\int_0^\ell R \phi_L(R) dR = \ell \int_0^\ell \phi_L(R) dR. \quad (\text{OC.20})$$

The only density that can satisfy equation (OC.20) is the Dirac delta $\phi_L(R) = \delta(R - \ell)$, which is not in Φ_L . Hence, for any $\phi_L \in \Phi_L$, there must exist some $\ell < r_* < k_*$ such that equation (OC.18) holds. Choosing r_* accordingly, the left-tail behavior of $\mathbb{Q}_t$ at any $t \in \mathcal{T} \setminus \{0\}$ is governed by the density $c_L(t) \phi_L(R)$ for $0 < R < r_*$, setting the normalizing constant $c_L(t) := P^{\mathbb{Q}\tau}(r_*, t) / \tilde{P}^{\phi_L}(r_*) > 0$.

For the convexity of $C^{\mathbb{Q}\tau}(k, t)$ at $k = r^*$, it similarly needs to hold that

$$\partial_{k+} C^{\mathbb{Q}\tau}(r^*, t) = C^{\mathbb{Q}\tau}(r^*, t) \frac{\partial_{k+} \tilde{C}^{\phi_R}(r^*)}{\tilde{C}^{\phi_R}(r^*)} \geq \partial_{k-} C^{\mathbb{Q}\tau}(r^*, t) = b_R(t). \quad (\text{OC.21})$$

Rewriting equation (OC.21) yields the equivalent condition

$$r^* - \frac{\tilde{C}^{\phi_R}(r^*)}{\partial_{k+} \tilde{C}^{\phi_R}(r^*)} = \frac{\int_{r^*}^\infty R \phi_R(R) dR}{\int_{r^*}^\infty \phi_R(R) dR} \geq r^* - \frac{C^{\mathbb{Q}\tau}(r^*, t)}{b_R(t)} = u. \quad (\text{OC.22})$$

Following an entirely symmetric argument as for the left tail, we may conclude that for any $\phi_R \in \Phi_R$, there must exist some $k^* < r^* < u$ such that equation (OC.22) holds. Using any such r^* , the right-tail behavior of $\mathbb{Q}_t$ at any $t \in \mathcal{T} \setminus \{0\}$ is governed by the density $c_R(t) \phi_R(R)$ for $R > r^*$, setting the normalizing constant $c_R(t) := C^{\mathbb{Q}\tau}(r^*, t) / \tilde{C}^{\phi_R}(r^*) > 0$. $\square$

OC.4 Relation of single-period and multi-period option price bounds

Proposition 3.1 states option price bounds within a multi-period setting under bid-ask constraints over a general strike-maturity grid $\mathbb{S}_{\mathbb{T}} := \mathbb{K} \times \mathbb{T}$. The resulting option price bounds generally satisfy the relation

$$\mathcal{L}^X(\kappa, \tau) = \bigvee_{t \geq \tau} \mathcal{L}^X(\kappa, t), \quad \mathcal{U}^X(\kappa, \tau) = \bigwedge_{t \geq \tau} \mathcal{U}^X(\kappa, t) \quad (\text{OC.23})$$

over $t \in \mathbb{T}$. Observing that $\mathcal{L}^\chi(\kappa, t)$ and $\mathcal{U}^\chi(\kappa, t)$ are monotonously increasing and decreasing in t , respectively, equation (OC.23) implies equation (3.12) when $\mathbb{T}$ is finite. Hence, the temporal behavior of the option price bounds for any $t \geq 0$ is fully characterized by the bounds at traded maturities $t \in \mathbb{T}$.

As a special case, Proposition 3.1 also yields option price bounds at any traded maturity $t \in \mathbb{T}$ in a single-period setting under bid-ask constraints over the strike-maturity grid $\mathbb{S}_t := \mathbb{K} \times \{t\}$, where the assumption of a maturity-independent strike grid $\mathbb{K}$ is without loss of generality. To further investigate their relations, we here denote $\mathcal{L}^\chi(\kappa, t; \mathbb{S}_t)$ and $\mathcal{U}^\chi(\kappa, t; \mathbb{S}_t)$ for the single-period bounds at a given $t \in \mathbb{T}$, while we use $\mathcal{L}^\chi(\kappa, \tau) := \mathcal{L}^\chi(\kappa, \tau; \mathbb{S}_{\mathbb{T}})$ and $\mathcal{U}^\chi(\kappa, \tau) := \mathcal{U}^\chi(\kappa, \tau; \mathbb{S}_{\mathbb{T}})$ for the multi-period bounds at any $\tau \geq 0$. As it turns out, the relation between the single-period and multi-period bounds is generally non-trivial.

In general, the upper bounds in Proposition 3.1 satisfy the relation

$$\mathcal{U}^\chi(\kappa, \tau; \mathbb{S}_{\mathbb{T}}) \leq \bigwedge_{t \geq \tau} \mathcal{U}^\chi(\kappa, t; \mathbb{S}_t) \quad (\text{OC.24})$$

over $t \in \mathbb{T}$. In particular, we thus have that $\mathcal{U}^\chi(\kappa, t; \mathbb{S}_{\mathbb{T}}) \leq \mathcal{U}^\chi(\kappa, t; \mathbb{S}_t)$ for any $t \in \mathbb{T}$. With regards to the lower bounds in Proposition 3.1, we generally have the analogous relation

$$\mathcal{L}^\chi(\kappa, \tau; \mathbb{S}_{\mathbb{T}}) \geq \bigvee_{t \leq \tau} \mathcal{L}^\chi(\kappa, t; \mathbb{S}_t) \quad (\text{OC.25})$$

over $t \in \mathbb{T}$. In particular, it therefore holds that $\mathcal{L}^\chi(\kappa, t; \mathbb{S}_{\mathbb{T}}) \geq \mathcal{L}^\chi(\kappa, t; \mathbb{S}_t)$ for any $t \in \mathbb{T}$.

Without further structure on the bid-ask constraints, the inequalities in relations (OC.24) and (OC.25) may be strict. The following corollary establishes simple conditions on ask prices that guarantee equality in both relations. For any $t \in \mathbb{T}$, equation (OC.26) in particular implies for the upper bound that $\mathcal{U}^\chi(\kappa, t; \mathbb{S}_{\mathbb{T}}) = \mathcal{U}^\chi(\kappa, t; \mathbb{S}_t)$. However, we do not obtain an equally simple equality for the lower bound $\mathcal{L}^\chi(\kappa, t; \mathbb{S}_{\mathbb{T}})$ from equation (OC.27), even when imposing analogous conditions on bid prices.

Corollary OC.3. *Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Moreover, suppose that $\mathcal{U}^C(k, t'; \mathbb{S}_{t'}) \leq A^C(k, t'')$ for all $k \in \mathbb{K}$ and $t', t'' \in \mathbb{T}$ with $t' \leq t''$. Then the following holds in problems (3.4) with $\kappa \geq 0$ and $\tau > 0$ for $t \in \mathbb{T}$:*

$$\mathcal{U}^\chi(\kappa, \tau) := \mathcal{U}^\chi(\kappa, \tau; \mathbb{S}_{\mathbb{T}}) = \bigwedge_{t \geq \tau} \mathcal{U}^\chi(\kappa, t; \mathbb{S}_t), \quad (\text{OC.26})$$

$$\mathcal{L}^\chi(\kappa, \tau) := \mathcal{L}^\chi(\kappa, \tau; \mathbb{S}_{\mathbb{T}}) = \bigvee_{t \leq \tau} \mathcal{L}^\chi(\kappa, t; \mathbb{S}_t). \quad (\text{OC.27})$$

Proof. Take any $\kappa \geq 0$ and $\tau > 0$. It suffices to verify equations (OC.26) and (OC.27) for the call price bounds in equation (3.6). The analogous statement for the put price bounds in equation (3.8) then follows immediately from the put-call parity relations (3.5).

For fixed $t', t'' \geq \tau$, denote the right-hand side of equation (3.6) by $U^C(\kappa, \tau; t', t'')$. Note that we thereby have $U^C(\kappa, \tau; t', t'') = U^C(\kappa, t' \wedge t''; t', t'')$. With this definition, we may express the upper bound $\mathcal{U}^C(\kappa, \tau) := \mathcal{U}^C(\kappa, \tau; \mathbb{S}_{\mathbb{T}})$ in equation (3.6) over $t', t'' \in \mathbb{T}$ as

$$\mathcal{U}^C(\kappa, \tau) = \bigwedge_{t', t'' \geq \tau} U^C(\kappa, \tau; t', t'') = \bigwedge_{t \geq \tau} \mathcal{U}^C(\kappa, t; \mathbb{S}_t) \wedge \bigwedge_{\substack{t', t'' \geq \tau \\ t' \neq t''}} U^C(\kappa, t' \wedge t''; t', t''), \quad (\text{OC.28})$$

using that $\mathcal{U}^C(\kappa, \tau; \mathbb{S}_t) = \mathcal{U}^C(\kappa, t; \mathbb{S}_t) = U^C(\kappa, t; t, t)$ for any $t \in \mathbb{T}$ with $t \geq \tau$.

Equation (OC.26) is established once we show that the term involving $U^C(\kappa, t' \wedge t''; t', t'')$ in equa-

tion (OC.28) is irrelevant. For this, setting $A^C(0, t) = 1$ and $A^C(\infty, t) = 0$ for all $t \in \mathbb{T}$, we exploit that for any $\varepsilon > 0$, there exist $k', k'' \in \mathbb{K} \cup \{0, \infty\}$ with $k' \leq \kappa < k''$ such that

$$\begin{aligned} U^C(\kappa, t' \wedge t''; t', t'') &\geq \frac{k'' - \kappa}{k'' - k'} A^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') - \varepsilon \\ &\geq \frac{k'' - \kappa}{k'' - k'} \mathcal{U}^C(k', t' \wedge t''; \mathbb{S}_{t' \wedge t''}) + \frac{\kappa - k'}{k'' - k'} \mathcal{U}^C(k'', t' \wedge t''; \mathbb{S}_{t' \wedge t''}) - \varepsilon \\ &\geq \mathcal{U}^C(\kappa, t' \wedge t''; \mathbb{S}_{t' \wedge t''}) - \varepsilon. \end{aligned}$$

Here, the first inequality follows by construction of $U^C(\kappa, \tau; t', t'')$ as an infimum, the second inequality holds by assumption, and the third inequality is due to the convexity of $\mathcal{U}^C(\cdot, t' \wedge t''; \mathbb{S}_{t' \wedge t''})$. Since the relation holds for any $\varepsilon > 0$, we conclude that $U^C(\kappa, t' \wedge t''; t', t'') \geq \mathcal{U}^C(\kappa, t' \wedge t''; \mathbb{S}_{t' \wedge t''})$, so that equation (OC.28) is equivalent to equation (OC.26).

For fixed $t' \leq \tau, t''$, analogously denote the right-hand side of equation (3.7) by $L^C(\kappa, \tau; t', t'')$. It immediately follows from this definition that $L^C(\kappa, \tau; t', t'') = L^C(\kappa, t'; t', t'')$. We may therefore write the lower bound $\mathcal{L}^C(\kappa, \tau) := \mathcal{L}^C(\kappa, \tau; \mathbb{S}_{\mathbb{T}})$ in equation (3.7) over $t', t'' \in \mathbb{T}$ as

$$\mathcal{L}^C(\kappa, \tau) = \bigvee_{t' \leq \tau} \bigvee_{t'' \geq t'} L^C(\kappa, \tau; t', t'') = \bigvee_{t \leq \tau} \mathcal{L}^C(\kappa, t; \mathbb{S}_t) \vee \bigvee_{t' \leq \tau} \bigvee_{t'' > t'} L^C(\kappa, t'; t', t''), \quad (\text{OC.29})$$

using that $\mathcal{L}^C(\kappa, \tau; \mathbb{S}_t) = \mathcal{L}^C(\kappa, t; \mathbb{S}_t) = L^C(\kappa, t; t, t)$ for any $t \in \mathbb{T}$ with $t \leq \tau$.

Equation (OC.29) implies equation (OC.27) if the term involving $L^C(\kappa, t'; t', t'')$ is irrelevant. Setting again $A^C(0, t) = 1$ and $A^C(\infty, t) = 0$ for all $t \in \mathbb{T}$, this can be verified noting that for any $\varepsilon > 0$, there exist $k', k'' \in \mathbb{K} \cup \{0, \infty\}$ with either $k'' < k' \leq \kappa$ or $\kappa \leq k' < k''$ such that

$$\begin{aligned} L^C(\kappa, t'; t', t'') &\leq \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} A^C(k'', t'') + \varepsilon \\ &\leq \frac{k'' - \kappa}{k'' - k'} B^C(k', t') + \frac{\kappa - k'}{k'' - k'} \mathcal{U}^C(k'', t'; \mathbb{S}_{t'}) + \varepsilon \\ &\leq \mathcal{L}^C(\kappa, t'; \mathbb{S}_{t'}) + \varepsilon. \end{aligned}$$

The first inequality results from the definition of $L^C(\kappa, t'; t', t'')$ as a supremum, the second inequality holds by assumption, and the third inequality is a consequence of the construction of the lower bound $\mathcal{L}^C(\kappa, t'; \mathbb{S}_{t'})$ and the convexity of $\mathcal{U}^C(\cdot, t'; \mathbb{S}_{t'})$. As the choice of $\varepsilon > 0$ is arbitrary, it therefore holds that $L^C(\kappa, t'; t', t'') \leq \mathcal{L}^C(\kappa, t'; \mathbb{S}_{t'})$. Hence, equation (OC.29) is equivalent to equation (OC.27). $\square$

While the relations established in Corollary OC.3 are useful, the required condition on ask prices may not hold in general. However, we can straightforwardly transform ask prices in such a way that an analogue to Corollary OC.3 holds without such a condition. To show this, we define a new set of bid and ask prices by pooling the original bid-ask prices across maturities $t' \in \mathbb{T}$ such that

$$\tilde{B}^C(k, t) := \sup\{B^C(k, t') : t' \leq t\}, \quad \tilde{A}^C(k, t) := \inf\{A^C(k, t') : t' \geq t\}$$

for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. Here, the pooling of bid prices is optional and only included for symmetry; we may as well simply define $\tilde{B}^C(k, t) := B^C(k, t)$. For the pooled bid-ask prices, analogous to equation (3.3), we then define the associated set of admissible risk-neutral probability measures

$$\tilde{\mathcal{O}}(\mathcal{T}) := \left\{ \mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T}) : \tilde{B}^C(k, t) \leq C^{\mathbb{Q}\tau}(k, t) \leq \tilde{A}^C(k, t), (k, t) \in \mathbb{K} \times \mathbb{T} \right\}. \quad (\text{OC.30})$$

Accordingly, we denote option price bounds over $\tilde{\mathcal{O}}(\mathcal{T})$, obtained from counterparts of problems (3.4), by $\tilde{\mathcal{L}}^\chi(\kappa, \tau)$ and $\tilde{\mathcal{U}}^\chi(\kappa, \tau)$.

Notably, it holds that $\tilde{\mathcal{O}}(\mathcal{T}) = \mathcal{O}(\mathcal{T})$, as the pooling is consistent with the convex order constraints in $\mathcal{Q}(\mathcal{T})$. Hence, we obtain that $\tilde{\mathcal{L}}^\chi(\kappa, \tau; \mathbb{S}_\mathbb{T}) = \mathcal{L}^\chi(\kappa, \tau; \mathbb{S}_\mathbb{T})$ and $\tilde{\mathcal{U}}^\chi(\kappa, \tau; \mathbb{S}_\mathbb{T}) = \mathcal{U}^\chi(\kappa, \tau; \mathbb{S}_\mathbb{T})$ in the notation employed here, so that the pooling of bid-ask prices is inconsequential for the multi-period option price bounds. However, the pooling may tighten single-period bounds as cross-maturity information is incorporated, such that we generally have $\tilde{\mathcal{L}}^\chi(\kappa, t; \mathbb{S}_t) \geq \mathcal{L}^\chi(\kappa, t; \mathbb{S}_t)$ and $\tilde{\mathcal{U}}^\chi(\kappa, t; \mathbb{S}_t) \leq \mathcal{U}^\chi(\kappa, t; \mathbb{S}_t)$. By construction, the pooled bid-ask prices satisfy the condition $\tilde{\mathcal{U}}^C(k, t'; \mathbb{S}_{t'}) \leq \tilde{\mathcal{A}}^C(k, t'')$ for all $k \in \mathbb{K}$ and $t', t'' \in \mathbb{T}$ with $t' \leq t''$. Therefore, Corollary OC.3 yields the following result, which conveys that single-period bounds based on pooled bid-ask constraints fully characterize the multi-period option price bounds in Proposition 3.1, without further conditions.

Corollary OC.4. *Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (3.4) with $\kappa \geq 0$ and $\tau > 0$ for $t \in \mathbb{T}$:*

$$\mathcal{U}^\chi(\kappa, \tau) := \mathcal{U}^\chi(\kappa, \tau; \mathbb{S}_\mathbb{T}) = \bigwedge_{t \geq \tau} \tilde{\mathcal{U}}^\chi(\kappa, t; \mathbb{S}_t), \quad (\text{OC.31})$$

$$\mathcal{L}^\chi(\kappa, \tau) := \mathcal{L}^\chi(\kappa, \tau; \mathbb{S}_\mathbb{T}) = \bigvee_{t \leq \tau} \tilde{\mathcal{L}}^\chi(\kappa, t; \mathbb{S}_t). \quad (\text{OC.32})$$

OC.5 Maturity-interpolated option price bounds

Suppose that the maturity set $\mathbb{T} \subset \mathcal{T}$ is finite. Similar to section 3, we use interpolation to further constrain $\mathcal{Q}(\mathcal{T})$. For this, we now consider a set $\mathcal{W}$ of possible interpolation weight functions $w(\tau; t', t'')$, where $t' := \max\{t \in \mathbb{T} \cup \{0\} : t \leq \tau\}$ and $t'' := \min\{t \in \mathbb{T} : t > \tau\}$, with the convention that $t'' := \infty$ whenever $\tau > \max \mathbb{T}$. For all $w \in \mathcal{W}$, we assume that $0 \leq w(\tau; t', t'') \leq 1$ with $w(t'; t', t'') = 1$ and that $w(\tau; t', t'')$ is monotonously decreasing in τ for fixed t', t'' . Given some $w \in \mathcal{W}$, we define the basic set of maturity-interpolated risk-neutral probability measures by

$$\bar{\mathcal{Q}}(\mathcal{T}; w) := \left\{ \mathbb{Q}_\mathcal{T} \in \mathcal{Q}(\mathcal{T}) : \mathbb{Q}_\tau = w(\tau; t', t'') \mathbb{Q}_{t'} + (1 - w(\tau; t', t'')) \mathbb{Q}_{t''} \text{ for all } \tau \notin \mathbb{T} \right\}.$$

Set $\bar{\mathcal{O}}(\mathcal{T}; w) := \mathcal{O}(\mathcal{T}) \cap \bar{\mathcal{Q}}(\mathcal{T}; w)$ for each $w \in \mathcal{W}$. Analogous to problems (3.4), we determine (optimal) lower and upper option price bounds over $\bar{\mathcal{O}}(\mathcal{T}; \mathcal{W}) := \bigcup_{w \in \mathcal{W}} \bar{\mathcal{O}}(\mathcal{T}; w)$ from the problems

$$\inf_{\mathbb{Q}_\mathcal{T} \in \bar{\mathcal{O}}(\mathcal{T}; \mathcal{W})} C^{\mathbb{Q}_\mathcal{T}}(\kappa, \tau), \quad \inf_{\mathbb{Q}_\mathcal{T} \in \bar{\mathcal{O}}(\mathcal{T}; \mathcal{W})} P^{\mathbb{Q}_\mathcal{T}}(\kappa, \tau). \quad (\text{OC.33})$$

The resulting lower and upper bounds are denoted by $\bar{\mathcal{L}}^\chi(\kappa, \tau)$ and $\bar{\mathcal{U}}^\chi(\kappa, \tau)$, respectively, and satisfy $\bar{\mathcal{L}}^\chi(\kappa, \tau) \geq \mathcal{L}^\chi(\kappa, \tau)$ and $\bar{\mathcal{U}}^\chi(\kappa, \tau) \leq \mathcal{U}^\chi(\kappa, \tau)$, with equality at every $\tau \in \mathbb{T}$.

The following corollary shows that the bounds $\bar{\mathcal{L}}^\chi(\kappa, \tau)$ and $\bar{\mathcal{U}}^\chi(\kappa, \tau)$ from problems (OC.33) correspond to certain maturity-interpolations of the option price bounds in Proposition 3.1 at maturities in $\mathbb{T}$.

Corollary OC.5. *Let $\mathbb{T} \subset \mathcal{T}$ be finite and $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then the following holds in problems (OC.33):*

$$\bar{\mathcal{U}}^\chi(\kappa, \tau) = w_*(\tau; t', t'') \mathcal{U}^\chi(\kappa, t') + (1 - w_*(\tau; t', t'')) \mathcal{U}^\chi(\kappa, t''), \quad (\text{OC.34})$$

$$\bar{\mathcal{L}}^\chi(\kappa, \tau) = w^*(\tau; t', t'') \mathcal{L}^\chi(\kappa, t') + (1 - w^*(\tau; t', t'')) \mathcal{L}^\chi(\kappa, t''), \quad (\text{OC.35})$$

where $t' := \max\{t \in \mathbb{T} \cup \{0\} : t \leq \tau\}$ and $t'' := \min\{t \in \mathbb{T} : t > \tau\}$ as well as

$$w_*(\tau; t', t'') := \inf_{w \in \mathcal{W}} w(\tau; t', t''), \quad w^*(\tau; t', t'') := \sup_{w \in \mathcal{W}} w(\tau; t', t'').$$

Proof. For each $w \in \mathcal{W}$, consider the auxiliary problems

$$\inf_{\mathbb{Q}_{\mathcal{T}} \in \overline{\mathcal{O}}(\mathcal{T}; w)} C^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau), \quad \inf_{\mathbb{Q}_{\mathcal{T}} \in \overline{\mathcal{O}}(\mathcal{T}; w)} P^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau). \quad (\text{OC.36})$$

Following the reasoning in the proof of Proposition 3.2, we find that the bounds in problems (OC.36) for all $w \in \mathcal{W}$ can be expressed as

$$\begin{aligned} \tilde{\mathcal{U}}^X(\kappa, \tau) &= w(\tau; t', t'') \mathcal{U}^X(\kappa, t') + (1 - w(\tau; t', t'')) \mathcal{U}^X(\kappa, t''), \\ \tilde{\mathcal{L}}^X(\kappa, \tau) &= w(\tau; t', t'') \mathcal{L}^X(\kappa, t') + (1 - w(\tau; t', t'')) \mathcal{L}^X(\kappa, t''). \end{aligned}$$

Optimizing pointwise over $w \in \mathcal{W}$ yields the bounds stated in equations (OC.34) and (OC.35). $\square$

Corollary OC.5 covers two important special cases. First, the expressions in Proposition 3.1 for any $\tau \in \mathcal{T}$ obtain when using for $\mathcal{W}$ the maximal set of all admissible interpolation weight functions. In that case, we have that $w_*(\tau; t', t'') = \mathbb{1}_{\{t'\}}(\tau)$ and $w^*(\tau; t', t'') = \mathbb{1}_{\{t'', t'\}}(\tau)$, which yields the relations in equation (3.12). Second, Proposition 3.2 corresponds to the case with singleton $\mathcal{W}$, containing only the linear interpolation weights $w(\tau; t', t'') = \frac{t'' - \tau}{t'' - t'}$.

OD Further results on probability bounds

This appendix provides further theoretical results regarding the probability bounds in section 3.3.

OD.1 Bounds for quantiles

The left-tail probabilities $LTP^{\mathbb{Q}_{\mathcal{T}}}(k, t)$, viewed as a function of $k \in \Omega$, yield the cumulative distribution function of R_t at time horizon $t \in \mathcal{T}$ under a specific $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$. Accordingly, section 3.3 provides model-free bounds for the cumulative distribution function.

To derive the corresponding bounds for quantiles, recall the definition of the α -quantile $Q^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t)$ associated to R_t at time horizon $t \in \mathcal{T}$ under a given $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$. Formally, $Q^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t)$ for $\alpha \in [0, 1]$ is any element of the interval $[Q_L^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t), Q_U^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t)]$, where the lower quantile $Q_L^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t)$ and the upper quantile $Q_U^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t)$ are defined in terms of left-tail probabilities (i.e., the cumulative distribution function) by

$$Q_L^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t) = \inf \left\{ k \geq 0 : LTP^{\mathbb{Q}_{\mathcal{T}}}(k, t) \geq \alpha \right\}, \quad (\text{OD.1})$$

$$Q_U^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t) = \sup \left\{ k \geq 0 : LTP^{\mathbb{Q}_{\mathcal{T}}}(k, t) \leq \alpha \right\}. \quad (\text{OD.2})$$

By construction, we have that $Q_L^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t) \leq Q_U^{\mathbb{Q}_{\mathcal{T}}}(\alpha, t)$.

Analogous to the option price and probability bounds in section 3, we may determine (optimal) lower and upper quantile bounds over $\mathcal{O}(\mathcal{T})$ at a given specification (α, τ) . For this purpose, we consider the problem

$$\inf_{\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}(\mathcal{T})} Q^{\mathbb{Q}_{\mathcal{T}}}(\alpha, \tau) \quad (\text{OD.3})$$

and denote the resulting lower bound (infimum) by $\mathcal{L}^Q(\alpha, \tau)$ and the upper bound (supremum) by $\mathcal{U}^Q(\alpha, \tau)$.

Provided feasibility (i.e., $\mathcal{O}(\mathcal{T}) \neq \emptyset$), the following proposition provides expressions for the quantile bounds in problem (OD.3) in terms of the left-tail probability bounds $\mathcal{L}^{LTP}(k, \tau)$ and $\mathcal{U}^{LTP}(k, \tau)$ in Proposition 3.3.

Proposition OD.1 (Model-free quantile bounds).

Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then in problem (OD.3), the bounds for quantiles are given by

$$\mathcal{U}^Q(\alpha, \tau) = \sup \left\{ k \geq 0 : \mathcal{L}^{LTP}(k, \tau) \leq \alpha \right\}, \quad (\text{OD.4})$$

$$\mathcal{L}^Q(\alpha, \tau) = \inf \left\{ k \geq 0 : \mathcal{U}^{LTP}(k, \tau) \geq \alpha \right\}. \quad (\text{OD.5})$$

Proof. The result directly follows from the definitions in equations (OD.1) and (OD.2).

Upper bound: To derive the expression in equation (OD.4), note that

$$\begin{aligned} \mathcal{U}^Q(\alpha, \tau) &= \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} Q_U^{\mathbb{Q}_\tau}(\alpha, \tau) = \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} \sup_{k \geq 0} \left\{ k \mathbb{1}_{[0, \alpha]}(LTP^{\mathbb{Q}_\tau}(k, \tau)) - \infty \mathbb{1}_{(\alpha, \infty)}(LTP^{\mathbb{Q}_\tau}(k, \tau)) \right\} \\ &= \sup_{k \geq 0} \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} \left\{ k \mathbb{1}_{[0, \alpha]}(LTP^{\mathbb{Q}_\tau}(k, \tau)) - \infty \mathbb{1}_{(\alpha, \infty)}(LTP^{\mathbb{Q}_\tau}(k, \tau)) \right\} \\ &= \sup_{k \geq 0} \left\{ k \mathbb{1}_{[0, \alpha]}(\mathcal{L}^{LTP}(k, \tau)) - \infty \mathbb{1}_{(\alpha, \infty)}(\mathcal{L}^{LTP}(k, \tau)) \right\} \\ &= \sup \left\{ k \geq 0 : \mathcal{L}^{LTP}(k, \tau) \leq \alpha \right\}, \end{aligned}$$

using the convention that $\infty \cdot 0 = 0$.

Lower bound: To derive the expression in equation (OD.5), similarly observe that

$$\begin{aligned} \mathcal{L}^Q(\alpha, \tau) &= \inf_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} Q_L^{\mathbb{Q}_\tau}(\alpha, \tau) = \inf_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} \inf_{k \geq 0} \left\{ k \mathbb{1}_{[\alpha, \infty)}(LTP^{\mathbb{Q}_\tau}(k, \tau)) + \infty \mathbb{1}_{[0, \alpha)}(LTP^{\mathbb{Q}_\tau}(k, \tau)) \right\} \\ &= \inf_{k \geq 0} \inf_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} \left\{ k \mathbb{1}_{[\alpha, \infty)}(LTP^{\mathbb{Q}_\tau}(k, \tau)) + \infty \mathbb{1}_{[0, \alpha)}(LTP^{\mathbb{Q}_\tau}(k, \tau)) \right\} \\ &= \inf_{k \geq 0} \left\{ k \mathbb{1}_{[\alpha, \infty)}(\mathcal{U}^{LTP}(k, \tau)) + \infty \mathbb{1}_{[0, \alpha)}(\mathcal{U}^{LTP}(k, \tau)) \right\} \\ &= \inf \left\{ k \geq 0 : \mathcal{U}^{LTP}(k, \tau) \geq \alpha \right\}, \end{aligned}$$

again using the convention that $\infty \cdot 0 = 0$. □

OD.2 Bounds for interval probabilities

For a given risk-neutral probability measure $\mathbb{Q}_\tau \in \mathcal{Q}(\mathcal{T})$, define the interval probability $IP^{\mathbb{Q}_\tau}(k_1, k_2, t)$ associated to R_t over the interval $[k_1, k_2] \subset \Omega$ with $k_1 < k_2$ and time horizon $t \in \mathcal{T}$ by

$$IP^{\mathbb{Q}_\tau}(k_1, k_2, t) := \mathbb{Q}_t(k_1 \leq R_t \leq k_2). \quad (\text{OD.6})$$

As in equation (2.1), we have that $IP^{\mathbb{Q}_\tau}(k_1, k_2, t) = \mathbb{E}^{\mathbb{Q}_t}[\mathbb{1}_{[k_1, k_2]}(R_t)] = \partial_{k+} C^{\mathbb{Q}_\tau}(k_2, t) - \partial_{k-} C^{\mathbb{Q}_\tau}(k_1, t)$. Economically, $IP^{\mathbb{Q}_\tau}(k_1, k_2, t)$ is the time-0 forward price of a European digital option on R_t . The tail probabilities in section 3.3 satisfy $LTP^{\mathbb{Q}_\tau}(k, t) = IP^{\mathbb{Q}_\tau}(0, k, t)$ and $RTP^{\mathbb{Q}_\tau}(k, t) = IP^{\mathbb{Q}_\tau}(k, \infty, t)$.

To quantify (optimal) lower and upper interval probability bounds over $\mathcal{O}(\mathcal{T})$ at a given specification

$(\kappa_1, \kappa_2, \tau)$, we consider the problem

$$\inf / \sup_{\mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T})} IP^{\mathbb{Q}_\tau}(\kappa_1, \kappa_2, \tau). \quad (\text{OD.7})$$

The resulting lower bound (infimum) is denoted by $\mathcal{L}^{IP}(\kappa_1, \kappa_2, \tau)$ and the upper bound (supremum) by $\mathcal{U}^{IP}(\kappa_1, \kappa_2, \tau)$. Trivially, we have that $\mathcal{L}^{IP}(\kappa_1, \kappa_2, 0) = \mathcal{U}^{IP}(\kappa_1, \kappa_2, 0) = \mathbb{1}_{[\kappa_1, \kappa_2]}(1)$.

In order to derive explicit expressions for the interval probability bounds in problem (OD.7), consider the auxiliary set of admissible risk-neutral probability measures

$$\mathcal{O}_{(a,b)}(\mathcal{T}) := \left\{ \mathbb{Q}_\tau \in \mathcal{O}(\mathcal{T}) : C^{\mathbb{Q}_\tau}(\kappa_1, \tau) = a, C^{\mathbb{Q}_\tau}(\kappa_2, \tau) = b \right\}, \quad (\text{OD.8})$$

which fixes the option prices at strikes κ_1 and κ_2 at given values a and b , respectively. Clearly, in light of Lemma OC.2, $\mathcal{O}_{(a,b)}(\mathcal{T}) \neq \emptyset$ holds whenever (a, b) is an element of the set

$$\mathcal{C}(\tau) := \left\{ (a, b) : \begin{aligned} & \mathcal{L}^C(\kappa_1, \tau) \leq a \leq \mathcal{U}^C(\kappa_1, \tau), \mathcal{L}^C(\kappa_2, \tau) \leq b \leq \mathcal{U}^C(\kappa_2, \tau), \\ & a > b, b < \mathcal{U}^C(k, \tau) \text{ for all } k < \kappa_2, b > \mathcal{L}^C(k, \tau) \text{ for all } k > \kappa_2, \\ & \frac{\kappa_2 - k}{\kappa_2 - \kappa_1} a + \frac{k - \kappa_1}{\kappa_2 - \kappa_1} b \geq \mathcal{L}^C(k, \tau) \text{ for all } \kappa_1 \leq k \leq \kappa_2, \\ & \frac{\kappa_2 - \kappa_1}{\kappa_2 - k} \mathcal{U}^C(k, \tau) + \frac{\kappa_1 - k}{\kappa_2 - k} b \geq a \text{ for all } k < \kappa_1, \\ & \frac{k - \kappa_2}{k - \kappa_1} a + \frac{\kappa_2 - \kappa_1}{k - \kappa_1} \mathcal{U}^C(k, \tau) \geq b \text{ for all } k > \kappa_2 \end{aligned} \right\}. \quad (\text{OD.9})$$

Each constraint set $\mathcal{O}_{(a,b)}(\mathcal{T})$ induces option price bounds as in Proposition 3.1, which will be denoted by $\mathcal{L}_{(a,b)}^C(k, t)$ and $\mathcal{U}_{(a,b)}^C(k, t)$ for $(k, t) \in \Omega \times \mathcal{T}$. Finally, note that $\mathcal{O}(\mathcal{T}) = \bigcup_{(a,b) \in \mathcal{C}(\tau)} \mathcal{O}_{(a,b)}(\mathcal{T})$.

Provided feasibility (i.e., $\mathcal{O}(\mathcal{T}) \neq \emptyset$), the following proposition states expressions for the interval probability bounds in problem (OD.7) in terms of the option price bounds $\mathcal{L}_{(a,b)}^C(k, \tau)$ and $\mathcal{U}_{(a,b)}^C(k, \tau)$. After some simplifications, Proposition 3.3 obtains as a special case for appropriate choices of κ_1 and κ_2 .

Proposition OD.2 (Model-free interval probability bounds).

Let $\mathcal{O}(\mathcal{T}) \neq \emptyset$. Then in problem (OD.7) for $k', k'' \in \mathbb{K} \cup \{0, 1, \kappa_1, \kappa_2\}$, the bounds for interval probabilities with $\kappa_2 > \kappa_1 \geq 0, \tau \geq 0$ are given by

$$\begin{aligned} \mathcal{U}^{IP}(\kappa_1, \kappa_2, \tau) = \sup_{(a,b) \in \mathcal{C}(\tau)} & \left(0 \wedge \inf \left\{ \frac{\mathcal{U}_{(a,b)}^C(k'', \tau) - \mathcal{L}_{(a,b)}^C(\kappa_2, \tau)}{k'' - \kappa_2} : \kappa_2 < k'' \right\} \right. \\ & \left. + \inf \left\{ -\frac{\mathcal{L}_{(a,b)}^C(\kappa_1, \tau) - \mathcal{U}_{(a,b)}^C(k', \tau)}{\kappa_1 - k'} : k' < \kappa_1 \right\} \right), \end{aligned} \quad (\text{OD.10})$$

$$\begin{aligned} \mathcal{L}^{IP}(\kappa_1, \kappa_2, \tau) = \inf_{(a,b) \in \mathcal{C}(\tau)} & \left(\sup \left\{ \lim_{\eta \downarrow 0} \frac{\mathcal{L}_{(a,b)}^C(\kappa_2 + \eta, \tau) - \mathcal{U}_{(a,b)}^C(k'', \tau)}{\kappa_2 + \eta - k''} : k'' \leq \kappa_2 \right\} \right. \\ & \left. + \sup \left\{ \lim_{\eta \downarrow 0} -\frac{\mathcal{U}_{(a,b)}^C(k', \tau) - \mathcal{L}_{(a,b)}^C(\kappa_1 - \eta, \tau)}{k' - (\kappa_1 - \eta)} : \kappa_1 \leq k' \right\} \right). \end{aligned} \quad (\text{OD.11})$$

Proof. In order to establish the stated expressions, consider for any $(a, b) \in \mathcal{C}(\tau)$ the auxiliary problem

$$\inf / \sup_{\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}_{(a,b)}(\mathcal{T})} IP^{\mathbb{Q}_{\mathcal{T}}}(\kappa_1, \kappa_2, \tau). \quad (\text{OD.12})$$

By construction, the bounds in problem (OD.7) obtain from the bounds in problem (OD.12) after optimizing over all $(a, b) \in \mathcal{C}(\tau)$. As the case $\tau = 0$ is trivial, the arguments will focus on $\tau > 0$.

Upper bound: Denote the right-hand side in equation (OD.10) by $U^{IP}(\kappa_1, \kappa_2, \tau)$ and the term in parentheses for given $(a, b) \in \mathcal{C}(\tau)$ accordingly by $U_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau)$. In fact, the latter corresponds to the respective supremum obtained from problem (OD.12). Given any $k' < \kappa_1 < \kappa_2 < k''$, combining equations (OB.16) and (OB.22) yields the "super-replicating" relation

$$\mathbb{I}_{[\kappa_1, \kappa_2]}(k) \leq \frac{(k - k')^+ - (k - \kappa_1)^+}{k' - \kappa_1} + \frac{(k - k'')^+ - (k - \kappa_2)^+}{k'' - \kappa_2}. \quad (\text{OD.13})$$

For each feasible $(a, b) \in \mathcal{C}(\tau)$, any admissible $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}_{(a,b)}(\mathcal{T})$ thus satisfies

$$\begin{aligned} IP^{\mathbb{Q}_{\mathcal{T}}}(\kappa_1, \kappa_2, \tau) \leq U_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau) := 0 \wedge \inf & \left\{ \frac{\mathcal{U}_{(a,b)}^C(k'', \tau) - \mathcal{L}_{(a,b)}^C(\kappa_2, \tau)}{k'' - \kappa_2} : \kappa_2 < k'' \right\} \\ & + \inf \left\{ -\frac{\mathcal{L}_{(a,b)}^C(\kappa_1, \tau) - \mathcal{U}_{(a,b)}^C(k', \tau)}{\kappa_1 - k'} : k' < \kappa_1 \right\} \end{aligned}$$

for $k', k'' \in \mathbb{K} \cup \{0\}$, so that $U_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau)$ is an upper bound over $\mathcal{O}_{(a,b)}(\mathcal{T})$.

Keep $(a, b) \in \mathcal{C}(\tau)$ fixed and take some large $u > \kappa_2$. Analogous to equations (OB.3) and (OB.8), define $U_{(a,b)}^C(k, t; u)$ and $L_{(a,b)}^C(k, t; u)$ for $(k, t) \in \Omega \times \mathcal{T}$ given an arbitrary $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}_{(a,b)}(\mathcal{T})$. Similar to equation (OB.9), define the set

$$\begin{aligned} \mathbb{L}_{\mathcal{T}}^{(a,b),u}(\kappa_1, \kappa_2, \tau) := & \{(k, t, A^C(k, t)) : (k, t) \in \mathbb{K} \times \mathbb{T}\} \cup \{(\kappa_1, \tau, a)\} \cup \{(\kappa_2, \tau, b)\} \\ & \cup \{(u', t, C^{\mathbb{Q}_{\mathcal{T}}^*}(u', t)) : u' \geq u, t \in \mathcal{T}\}. \end{aligned} \quad (\text{OD.14})$$

Following the usual reasoning, the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^{(a,b),u}(\kappa_1, \kappa_2, \tau))$ is associated to a probability measure $\mathbb{Q}_{\mathcal{T}}^u \in \mathcal{O}_{(a,b)}(\mathcal{T})$, while we have that $\mathcal{E}(\kappa_1, \tau; \mathbb{L}_{\mathcal{T}}^{(a,b),u}(\kappa_1, \kappa_2, \tau)) = L_{(a,b)}^C(\kappa_1, \tau; u) = a$ and $\mathcal{E}(\kappa_2, \tau; \mathbb{L}_{\mathcal{T}}^{(a,b),u}(\kappa_1, \kappa_2, \tau)) = L_{(a,b)}^C(\kappa_2, \tau; u) = b$. By construction,

$$\begin{aligned} IP_{(a,b)}^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa_1, \kappa_2, \tau) &= \partial_{k+} \mathcal{E}(\kappa_2, \tau; \mathbb{L}_{\mathcal{T}}^{(a,b),u}(\kappa_1, \kappa_2, \tau)) - \partial_{k-} \mathcal{E}(\kappa_1, \tau; \mathbb{L}_{\mathcal{T}}^{(a,b),u}(\kappa_1, \kappa_2, \tau)) \\ &= \inf \left\{ \frac{U_{(a,b)}^C(k'', \tau; u) - L_{(a,b)}^C(\kappa_2, \tau; u)}{k'' - \kappa_2} : \kappa_2 < k'' \right\} \\ &\quad + \inf \left\{ -\frac{L_{(a,b)}^C(\kappa_1, \tau; u) - U_{(a,b)}^C(k', \tau; u)}{\kappa_1 - k'} : k' < \kappa_1 \right\}. \end{aligned}$$

With an analogous reasoning as in the proof of Proposition 3.3, we find that $IP_{(a,b)}^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa_1, \kappa_2, \tau) \rightarrow U_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau)$ as $u \rightarrow \infty$. Hence, for some appropriate sequence of (a, b) in $\mathcal{C}(\tau)$, it also follows that

$$IP_{(a,b)}^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa_1, \kappa_2, \tau) \longrightarrow U^{IP}(\kappa_1, \kappa_2, \tau) := \sup_{(a,b) \in \mathcal{C}(\tau)} U_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau),$$

such that $U^{IP}(\kappa_1, \kappa_2, \tau)$ must coincide with the supremum $\mathcal{U}^{IP}(\kappa_1, \kappa_2, \tau)$ in problem (OD.7).

Lower bound: Denote the right-hand side in equation (OD.11) by $L^{IP}(\kappa_1, \kappa_2, \tau)$ and the term in parentheses for given $(a, b) \in \mathcal{C}(\tau)$ by $L_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau)$. The latter corresponds to the respective infimum obtained from problem (OD.12). For any $\kappa_1 \leq k', k'' \leq \kappa_2$, combining equations (OB.19) and (OB.25) yields the "sub-replicating" relation

$$\mathbb{I}_{[\kappa_1, \kappa_2]}(k) \geq \lim_{\tilde{k} \uparrow \kappa_1} -\frac{(k - k')^+ - (k - \tilde{k})^+}{k' - \tilde{k}} + \lim_{\tilde{k} \downarrow \kappa_2} \frac{(k - \tilde{k})^+ - (k - k'')^+}{\tilde{k} - k''}. \quad (\text{OD.15})$$

Given any feasible $(a, b) \in \mathcal{C}(\tau)$, it therefore holds for each admissible $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}_{(a,b)}(\mathcal{T})$ that

$$\begin{aligned} IP^{\mathbb{Q}_{\mathcal{T}}}(\kappa_1, \kappa_2, \tau) &\geq L_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau) := \sup \left\{ \lim_{\eta \downarrow 0} \frac{\mathcal{L}_{(a,b)}^C(\kappa_2 + \eta, \tau) - \mathcal{U}_{(a,b)}^C(k'', \tau)}{\kappa_2 + \eta - k''} : k'' \leq \kappa_2 \right\} \\ &\quad + \sup \left\{ \lim_{\eta \downarrow 0} -\frac{\mathcal{U}_{(a,b)}^C(k', \tau) - \mathcal{L}_{(a,b)}^C(\kappa_1 - \eta, \tau)}{k' - (\kappa_1 - \eta)} : \kappa_1 \leq k' \right\} \end{aligned}$$

for $k', k'' \in \mathbb{K} \cup \{\kappa_1, \kappa_2\}$, implying that $L_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau)$ is a lower bound over $\mathcal{O}_{(a,b)}(\mathcal{T})$.

Fix $(a, b) \in \mathcal{C}(\tau)$ and take some large $u > \kappa_2$ and small $\eta > 0$. Similar to equation (OD.14), define the set

$$\begin{aligned} \mathbb{L}_{\mathcal{T}}^{(a,b),u,\eta}(\kappa_1, \kappa_2, \tau) &:= \{(k, t, A^C(k, t)) : (k, t) \in \mathbb{K} \times \mathbb{T}\} \cup \{(\kappa_1, \tau, a)\} \cup \{(\kappa_2, \tau, b)\} \\ &\quad \cup \{(\kappa_1 - \eta, \tau, L_{(a,b)}^C(\kappa_1 - \eta, \tau; u))\} \cup \{(\kappa_2 + \eta, \tau, L_{(a,b)}^C(\kappa_2 + \eta, \tau; u))\} \\ &\quad \cup \{(u', t, C^{\mathbb{Q}_{\mathcal{T}}}(u', t)) : u' \geq u, t \in \mathcal{T}\}. \end{aligned} \quad (\text{OD.16})$$

Consider the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^{(a,b),u,\eta}(\kappa_1, \kappa_2, \tau))$, which by construction is associated to a probability measure $\mathbb{Q}_{\mathcal{T}}^{u,\eta} \in \mathcal{O}_{(a,b)}(\mathcal{T})$. It thereby holds that

$$\begin{aligned} IP_{(a,b)}^{\mathbb{Q}_{\mathcal{T}}^{u,\eta}}(\kappa_1, \kappa_2, \tau) &= \partial_{k+} \mathcal{E}(\kappa_2, \tau; \mathbb{L}_{\mathcal{T}}^{(a,b),u,\eta}(\kappa_1, \kappa_2, \tau)) - \partial_{k-} \mathcal{E}(\kappa_1, \tau; \mathbb{L}_{\mathcal{T}}^{(a,b),u,\eta}(\kappa_1, \kappa_2, \tau)) \\ &\leq \sup \left\{ \frac{L_{(a,b)}^C(\kappa_2 + \eta, \tau; u) - U_{(a,b)}^C(k'', \tau; u)}{\kappa_2 + \eta - k''} : k'' < \kappa_2 + \eta \right\} \\ &\quad + \sup \left\{ -\frac{U_{(a,b)}^C(k', \tau; u) - L_{(a,b)}^C(\kappa_1 - \eta, \tau; u)}{k' - (\kappa_1 - \eta)} : \kappa_1 - \eta < k' \right\}. \end{aligned}$$

Following an analogous reasoning as in the proof of Proposition 3.3, we have that $IP_{(a,b)}^{\mathbb{Q}_{\mathcal{T}}^{u,\eta}}(\kappa_1, \kappa_2, \tau) \rightarrow L_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau)$ as $u \rightarrow \infty$ and $\eta \rightarrow 0$. Therefore, an appropriate sequence of (a, b) in $\mathcal{C}(\tau)$ further yields that

$$IP_{(a,b)}^{\mathbb{Q}_{\mathcal{T}}^{u,\eta}}(\kappa_1, \kappa_2, \tau) \longrightarrow L^{IP}(\kappa_1, \kappa_2, \tau) := \inf_{(a,b) \in \mathcal{C}(\tau)} L_{(a,b)}^{IP}(\kappa_1, \kappa_2, \tau),$$

implying that $L^{IP}(\kappa_1, \kappa_2, \tau)$ must coincide with the infimum $\mathcal{L}^{IP}(\kappa_1, \kappa_2, \tau)$ in problem (OD.7). $\square$

OD.3 Bounds for point probabilities

As a limiting case of the interval probabilities in section OD.2, consider for a given risk-neutral probability measure $\mathbb{Q}_{\mathcal{T}} \in \mathcal{Q}(\mathcal{T})$ the point probability $PP^{\mathbb{Q}_{\mathcal{T}}}(k, t)$ associated to R_t at point $k \in \Omega$ and time horizon $t \in \mathcal{T}$, defined by

$$PP^{\mathbb{Q}_{\mathcal{T}}}(k, t) := \mathbb{Q}_t(R_t = k). \quad (\text{OD.17})$$

By equation (2.1), it holds that $PP^{\mathbb{Q}\tau}(k, t) = \mathbb{E}^{\mathbb{Q}_t}[\mathbb{I}_{\{k\}}(R_t)] = \partial_{k+} C^{\mathbb{Q}\tau}(k, t) - \partial_{k-} C^{\mathbb{Q}\tau}(k, t)$.

As usual, we determine (optimal) lower and upper point probability bounds at a given specification (κ, τ) from the problem

$$\inf / \sup_{\mathbb{Q}\tau \in \mathcal{O}(\tau)} PP^{\mathbb{Q}\tau}(\kappa, \tau). \quad (\text{OD.18})$$

The resulting lower bound (infimum) is denoted by $\mathcal{L}^{PP}(\kappa, \tau)$ and the upper bound (supremum) by $\mathcal{U}^{PP}(\kappa, \tau)$. The trivial identity $\mathcal{L}^{PP}(\kappa, 0) = \mathcal{U}^{PP}(\kappa, 0) = \mathbb{I}_{\{\kappa\}}(1)$ holds.

Similar to section OD.2, consider for a given κ the auxiliary set of admissible risk-neutral probability measures

$$\mathcal{O}_{(a)}(\tau) := \left\{ \mathbb{Q}\tau \in \mathcal{O}(\tau) : C^{\mathbb{Q}\tau}(\kappa, \tau) = a \right\}, \quad (\text{OD.19})$$

where we now have that $\mathcal{O}_{(a)}(\tau) \neq \emptyset$ exactly if a is an element of the set

$$\mathcal{A}(\tau) := \left\{ a : \begin{array}{l} \mathcal{L}^C(\kappa, \tau) \leq a \leq \mathcal{U}^C(\kappa, \tau), \\ a < \mathcal{U}^C(k, \tau) \text{ for all } k < \kappa, \ a > \mathcal{L}^C(k, \tau) \text{ for all } k > \kappa \end{array} \right\}. \quad (\text{OD.20})$$

Each constraint set $\mathcal{O}_{(a)}(\tau)$ likewise induces option price bounds as in Proposition 3.1, which are denoted by $\mathcal{L}_{(a)}^C(k, t)$ and $\mathcal{U}_{(a)}^C(k, t)$ for $(k, t) \in \Omega \times \mathcal{T}$. Analogous to before, note that $\mathcal{O}(\tau) = \bigcup_{a \in \mathcal{A}(\tau)} \mathcal{O}_{(a)}(\tau)$.

Provided feasibility (i.e., $\mathcal{O}(\tau) \neq \emptyset$), the following proposition states explicit expressions for the point probability bounds in problem (OD.18) in terms of the option price bounds $\mathcal{L}^C(k, \tau)$ and $\mathcal{U}^C(k, \tau)$.

Proposition OD.3 (Model-free point probability bounds).

Let $\mathcal{O}(\tau) \neq \emptyset$. Then in problem (OD.18) for $k', k'' \in \mathbb{K} \cup \{0, 1, \kappa\}$, the bounds for point probabilities with $\kappa \geq 0, \tau \geq 0$ are given by

$$\mathcal{U}^{PP}(\kappa, \tau) = 0 \wedge \inf \left\{ \frac{\mathcal{U}^C(k'', \tau) - \mathcal{L}^C(\kappa, \tau)}{k'' - \kappa} : \kappa < k'' \right\} + \inf \left\{ -\frac{\mathcal{L}^C(\kappa, \tau) - \mathcal{U}^C(k', \tau)}{\kappa - k'} : k' < \kappa \right\}, \quad (\text{OD.21})$$

$$\mathcal{L}^{PP}(\kappa, \tau) = \begin{cases} 0 \vee \left(\partial_{k+} \mathcal{L}^C(\kappa, \tau) - \partial_{k-} \mathcal{L}^C(\kappa, \tau) \right) & \text{if } \mathcal{L}^C(\kappa, \tau) = \mathcal{U}^C(\kappa, \tau) \\ 0 & \text{if } \mathcal{L}^C(\kappa, \tau) < \mathcal{U}^C(\kappa, \tau) \end{cases}. \quad (\text{OD.22})$$

Proof. For any $a \in \mathcal{A}(\tau)$, now consider the auxiliary problem

$$\inf / \sup_{\mathbb{Q}\tau \in \mathcal{O}_{(a)}(\tau)} PP^{\mathbb{Q}\tau}(\kappa, \tau). \quad (\text{OD.23})$$

By construction, the bounds in problem (OD.18) obtain from the bounds in problem (OD.23) after optimizing over all $a \in \mathcal{A}(\tau)$. The arguments will again focus on the non-trivial case $\tau > 0$.

Upper bound: The relevant "super-replicating" relations for $k' < \kappa < k''$ obtain from equation (OD.13) when setting $\kappa_1 = \kappa_2 = \kappa$. Given a feasible $a \in \mathcal{A}(\tau)$, every admissible $\mathbb{Q}\tau \in \mathcal{O}_{(a)}(\tau)$ thus satisfies

$$\begin{aligned} PP^{\mathbb{Q}\tau}(\kappa, \tau) &\leq U_{(a)}^{PP}(\kappa, \tau) := 0 \wedge \inf \left\{ \frac{\mathcal{U}_{(a)}^C(k'', \tau) - \mathcal{L}_{(a)}^C(\kappa, \tau)}{k'' - \kappa} : \kappa < k'' \right\} \\ &\quad + \inf \left\{ -\frac{\mathcal{L}_{(a)}^C(\kappa, \tau) - \mathcal{U}_{(a)}^C(k', \tau)}{\kappa - k'} : k' < \kappa \right\} \end{aligned}$$

for $k', k'' \in \mathbb{K} \cup \{0\}$. Hence, $U_{(a)}^{PP}(\kappa, \tau)$ is an upper bound over $\mathcal{O}_{(a)}(\mathcal{T})$.

Fix $a \in \mathcal{A}(\tau)$ and take some large $u > \kappa$. Analogous to equations (OB.3) and (OB.8), now define $U_{(a)}^C(k, t; u)$ and $L_{(a)}^C(k, t; u)$ for $(k, t) \in \Omega \times \mathcal{T}$ and an arbitrary $\mathbb{Q}_{\mathcal{T}}^* \in \mathcal{O}_{(a)}(\mathcal{T})$. Adjusting the definition in equation (OD.14), take the set

$$\begin{aligned} \mathbb{L}_{\mathcal{T}}^{(a),u}(\kappa, \tau) &:= \{(k, t, A^C(k, t)) : (k, t) \in \mathbb{K} \times \mathbb{T}\} \cup \{(\kappa, \tau, a)\} \\ &\cup \{(u', t, C^{\mathbb{Q}_{\mathcal{T}}^*}(u', t)) : u' \geq u, t \in \mathcal{T}\}. \end{aligned} \quad (\text{OD.24})$$

Consider the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^{(a),u}(\kappa, \tau))$, which is associated to a probability measure $\mathbb{Q}_{\mathcal{T}}^u \in \mathcal{O}_{(a)}(\mathcal{T})$ and satisfies $\mathcal{E}(\kappa, \tau; \mathbb{L}_{\mathcal{T}}^{(a),u}(\kappa, \tau)) = L_{(a)}^C(\kappa, \tau; u) = a$. We then have that

$$\begin{aligned} PP_{(a)}^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, \tau) &= \partial_{k+} \mathcal{E}(\kappa, \tau; \mathbb{L}_{\mathcal{T}}^{(a),u}(\kappa, \tau)) - \partial_{k-} \mathcal{E}(\kappa, \tau; \mathbb{L}_{\mathcal{T}}^{(a),u}(\kappa, \tau)) \\ &= \inf \left\{ \frac{U_{(a)}^C(k'', \tau; u) - L_{(a)}^C(\kappa, \tau; u)}{k'' - \kappa} : \kappa < k'' \right\} \\ &\quad + \inf \left\{ -\frac{L_{(a)}^C(\kappa, \tau; u) - U_{(a)}^C(k', \tau; u)}{\kappa - k'} : k' < \kappa \right\}. \end{aligned}$$

An analogous reasoning as before justifies that $PP_{(a)}^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, \tau) \rightarrow U_{(a)}^{PP}(\kappa, \tau)$ as $u \rightarrow \infty$. For an appropriate sequence of a in $\mathcal{A}(\tau)$, we thus obtain

$$PP_{(a)}^{\mathbb{Q}_{\mathcal{T}}^u}(\kappa, \tau) \longrightarrow U^{PP}(\kappa, \tau) := \sup_{a \in \mathcal{A}(\tau)} U_{(a)}^{PP}(\kappa, \tau).$$

Hence, $U^{PP}(\kappa, \tau)$ equals the supremum $\mathcal{U}^{PP}(\kappa, \tau)$ in problem (OD.18).

It remains to show that $U^{PP}(\kappa, \tau)$ is equivalent to the right-hand side in equation (OD.21). For this, observe that $U_{(a)}^{PP}(\kappa, \tau)$ is monotonously decreasing in a and the right-hand side in equation (OD.21) coincides with $U_{(a)}^{PP}(\kappa, \tau)$ for the optimal choice $a \rightarrow \inf \mathcal{A}(\tau) = \mathcal{L}^C(\kappa, \tau)$.

Lower bound: Equation (OD.15) with $\kappa_1 = \kappa_2 = \kappa$ together with the trivial inequality $\mathbb{1}_{\{\kappa\}}(k) \geq 0$ yields the relevant "sub-replicating" relations. Accordingly, each admissible $\mathbb{Q}_{\mathcal{T}} \in \mathcal{O}_{(a)}(\mathcal{T})$ given some feasible $a \in \mathcal{A}(\tau)$ satisfies

$$\begin{aligned} PP^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau) &\geq L_{(a)}^{PP}(\kappa, \tau) := 0 \vee \left(\lim_{\eta \downarrow 0} \frac{\mathcal{L}_{(a)}^C(\kappa + \eta, \tau) - \mathcal{U}_{(a)}^C(\kappa, \tau)}{\eta} \right. \\ &\quad \left. - \lim_{\eta \downarrow 0} \frac{\mathcal{U}_{(a)}^C(\kappa, \tau) - \mathcal{L}_{(a)}^C(\kappa - \eta, \tau)}{\eta} \right), \end{aligned} \quad (\text{OD.25})$$

yielding that $L_{(a)}^{PP}(\kappa, \tau)$ is a lower bound over $\mathcal{O}_{(a)}(\mathcal{T})$. Using that $\mathcal{L}_{(a)}^C(\kappa, \tau) = \mathcal{U}_{(a)}^C(\kappa, \tau) = a$ by construction, equation (OD.25) can be rewritten as

$$PP^{\mathbb{Q}_{\mathcal{T}}}(\kappa, \tau) \geq L_{(a)}^{PP}(\kappa, \tau) := 0 \vee \left(\partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau) - \partial_{k-} \mathcal{L}_{(a)}^C(\kappa, \tau) \right). \quad (\text{OD.26})$$

Now consider two different cases. First, suppose that $\mathcal{L}^C(\kappa, \tau) < \mathcal{U}^C(\kappa, \tau)$. Take any a strictly between these bounds, i.e., in the interior of $\mathcal{A}(\tau)$. Then $\partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau) - \partial_{k-} \mathcal{L}_{(a)}^C(\kappa, \tau) < 0$ by construction of the

option price bounds in Proposition 3.1 and thus $L_{(a)}^{PP}(\kappa, \tau) = 0$. Hence, in this case,

$$L^{PP}(\kappa, \tau) := \inf_{a \in \mathcal{A}(\tau)} L_{(a)}^{PP}(\kappa, \tau) = 0.$$

For some large enough $u > \kappa$ and small enough $\eta > 0$, the bound is attained by a probability measure $\mathbb{Q}_{\mathcal{T}}^{u, \eta} \in \mathcal{O}_{(a)}(\mathcal{T})$ associated to the lower convex envelope $\mathcal{E}(\cdot, \cdot; \mathbb{L}_{\mathcal{T}}^{(a', b'), u, \eta}(\kappa, \tau))$. Here, augmenting equation (OD.24), we employ the definition

$$\begin{aligned} \tilde{\mathbb{L}}_{\mathcal{T}}^{(a), u, \eta}(\kappa, \tau) := & \{(k, t, A^C(k, t)) : (k, t) \in \mathbb{K} \times \mathbb{T}\} \cup \{(\kappa, \tau, a)\} \\ & \cup \{(\kappa - \eta, \tau, a')\} \cup \{(\kappa + \eta, \tau, b')\} \cup \{(u', t, C^{\mathbb{Q}_{\mathcal{T}}^*(u', t)} : u' \geq u, t \in \mathcal{T}\}. \end{aligned} \quad (\text{OD.27})$$

with $a' := a - \eta \partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau)$ and $b' := a + \eta \partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau)$. Therefore, $L^{PP}(\kappa, \tau)$ equals the infimum $\mathcal{L}^{PP}(\kappa, \tau)$ in problem (OD.18), consistent with the respective case in equation (OD.22).

Second, suppose that $\mathcal{L}^C(\kappa, \tau) = \mathcal{U}^C(\kappa, \tau)$. For a equal to these bounds, take again some large enough $u > \kappa$ and small enough $\eta > 0$. Now set $a' := (a - \eta \partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau)) \vee L_{(a)}^C(\kappa - \eta, \tau; u)$ and $b' := (a + \eta \partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau)) \vee L_{(a)}^C(\kappa + \eta, \tau; u)$. The lower convex envelope $\mathcal{E}(\cdot, \cdot; \tilde{\mathbb{L}}_{\mathcal{T}}^{(a), u, \eta}(\kappa, \tau))$ is then associated to a probability measure $\mathbb{Q}_{\mathcal{T}}^{u, \eta} \in \mathcal{O}_{(a)}(\mathcal{T})$ such that

$$\begin{aligned} PP_{(a)}^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) &= \partial_{k+} \mathcal{E}(\kappa, \tau; \tilde{\mathbb{L}}_{\mathcal{T}}^{(a), u, \eta}(\kappa, \tau)) - \partial_{k-} \mathcal{E}(\kappa, \tau; \tilde{\mathbb{L}}_{\mathcal{T}}^{(a), u, \eta}(\kappa, \tau)) \\ &\leq \sup \left\{ \frac{L_{(a)}^C(\kappa + \eta, \tau; u) - U_{(a)}^C(k'', \tau; u)}{\kappa + \eta - k''} : k'' < \kappa + \eta \right\} \vee \left(\partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau) \right) \\ &\quad + \sup \left\{ -\frac{U_{(a)}^C(k', \tau; u) - L_{(a)}^C(\kappa - \eta, \tau; u)}{k' - (\kappa - \eta)} : \kappa_1 - \eta < k' \right\} \vee \left(-\partial_{k+} \mathcal{L}_{(a)}^C(\kappa, \tau) \right). \end{aligned}$$

By a similar reasoning as before, it holds that $PP_{(a)}^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) \rightarrow L_{(a)}^{PP}(\kappa, \tau)$ as $u \rightarrow \infty$ and $\eta \rightarrow 0$. Hence, since $\mathcal{A}(\tau) = \{a\}$ is a singleton in this case, we also have that

$$PP_{(a)}^{\mathbb{Q}_{\mathcal{T}}^{u, \eta}}(\kappa, \tau) \longrightarrow L^{PP}(\kappa, \tau) := \inf_{a \in \mathcal{A}(\tau)} L_{(a)}^{PP}(\kappa, \tau).$$

As a consequence, $L^{PP}(\kappa, \tau)$ likewise coincides with the infimum $\mathcal{L}^{PP}(\kappa, \tau)$ in problem (OD.18), consistent with the respective case in equation (OD.22). $\square$

OE Examples

This section illustrates several special cases of the option price bounds in Proposition 3.1 and tail probability bounds in Proposition 3.3.

OE.1 Continuous strikes, single maturity, without bid-ask spreads

Analogous to the main part in Bondarenko et al. (2024), suppose that option prices without bid-ask spreads are observed at a single maturity $\mathbb{T} := \{t\}$ with $t > 0$ and for every strike in a bounded interval $\mathbb{K} := [k_*, k^*] \subset (0, \bar{K})$, where $k_* < 1 < k^*$. Bid and ask prices at any $k \in \mathbb{K}$ are given by $B^C(k, t) = A^C(k, t) = C(k)$, where $C(k)$ is some function that represents arbitrage-free call prices in accordance with Lemma OA.2 and $P(k) = C(k) + k - 1$ yields the associated put prices.

Option price bounds

For the option price bounds in Proposition 3.1 at maturity $\tau > 0$, it holds that $\mathcal{U}^\chi(\kappa, \tau) = \mathcal{U}^\chi(\kappa, t)$ if $0 < \tau \leq t$ and $\mathcal{U}^C(\kappa, \tau) = 1$, $\mathcal{U}^P(\kappa, \tau) = \kappa$ if $\tau > t$ as well as $\mathcal{L}^C(\kappa, \tau) = (1 - \kappa)^+$, $\mathcal{L}^P(\kappa, \tau) = (\kappa - 1)^+$ if $0 < \tau < t$ and $\mathcal{L}^\chi(\kappa, \tau) = \mathcal{L}^\chi(\kappa, t)$ if $\tau \geq t$. Hence, it suffices to characterize the bounds at $\tau = t$, for which we obtain simple, tractable expressions in this case.

The call price bounds in equations (3.6) and (3.7) evaluated at $\tau = t$ for any $\kappa \geq 0$ can be expressed in closed form as

$$\begin{aligned}\mathcal{U}^C(\kappa, t) &= C((\kappa \vee k_*) \wedge k^*) - \frac{C(k_*) - 1}{k_*} (k_* - \kappa)^+, \\ \mathcal{L}^C(\kappa, t) &= (1 - \kappa)^+ \vee (C((\kappa \vee k_*) \wedge k^*) - \partial_{k+} C(k_*) (k_* - \kappa)^+ + \partial_{k-} C(k^*) (\kappa - k^*)^+).\end{aligned}$$

Likewise, the put price bounds in equations (3.8) and (3.9) at $\tau = t$ for any $\kappa \geq 0$ can be written in closed form as

$$\begin{aligned}\mathcal{U}^P(\kappa, t) &= P((\kappa \vee k_*) \wedge k^*) - \frac{P(k_*)}{k_*} (k_* - \kappa)^+ + (\kappa - k^*)^+, \\ \mathcal{L}^P(\kappa, t) &= (\kappa - 1)^+ \vee (P((\kappa \vee k_*) \wedge k^*) - \partial_{k+} P(k_*) (k_* - \kappa)^+ + \partial_{k-} P(k^*) (\kappa - k^*)^+).\end{aligned}$$

In this setting, we also obtain simple expressions for $z_*(t)$ and $z^*(t)$ as in Lemma OC.5, which determine the strike interval on which the lower bound $\mathcal{L}^P(\cdot, t)$ is non-trivial. Concretely, we find that

$$z_*(t) = 1 \wedge \left(\tilde{z}_*(t) - \frac{P(\tilde{z}_*(t))}{\partial_{k+} P(\tilde{z}_*(t))} \right), \quad z^*(t) = 1 \vee \left(\tilde{z}^*(t) - \frac{C(\tilde{z}^*(t))}{\partial_{k-} C(\tilde{z}^*(t))} \right),$$

where $\tilde{z}_*(t) = 1 \wedge \min\{k \in \mathbb{K} : P(k) > (k - 1)^+\}$ and $\tilde{z}^*(t) = 1 \wedge \max\{k \in \mathbb{K} : C(k) > (1 - k)^+\}$. Obviously, $\tilde{z}_*(\tau) = k_*$ holds if $P(k_*) > 0$ and $\tilde{z}^*(\tau) = k^*$ if $C(k^*) > 0$. Except for pathological cases, we generally have that $z_*(t) > 0$.

The expressions for $z_*(t)$ and $z^*(t)$ allow to rewrite the option price bounds in explicit piecewise-functional form. To arrive at the following expressions, we suppose without loss of generality that $z_*(t) \leq k_*$ and $z^*(t) \geq k^*$ (as the observed strike interval in the present setting can always be truncated to $[k_* \vee z_*(t), k^* \wedge z^*(t)]$). The call price bounds for $\tau = t$ and $\kappa \geq 0$ may then be written as

$$\begin{aligned}\mathcal{U}^C(\kappa, t) &= \mathbb{1}_{\{\kappa < k_*\}} \left(C(k_*) + \frac{C(k_*) - 1}{k_*} (\kappa - k_*) \right) + \mathbb{1}_{\{k_* \leq \kappa \leq k^*\}} C(\kappa) + \mathbb{1}_{\{k^* < \kappa\}} C(k^*), \\ \mathcal{L}^C(\kappa, t) &= \mathbb{1}_{\{\kappa \leq z_*(t)\}} (1 - \kappa) + \mathbb{1}_{\{z_*(t) < \kappa < k_*\}} (C(k_*) + \partial_{k+} C(k_*) (\kappa - k_*)) \\ &\quad + \mathbb{1}_{\{k_* \leq \kappa \leq k^*\}} C(\kappa) + \mathbb{1}_{\{k^* < \kappa < z^*(t)\}} (C(k^*) + \partial_{k-} C(k^*) (\kappa - k^*)).\end{aligned}$$

Moreover, the put price bounds for $\tau = t$ and $\kappa \geq 0$ can be expressed as

$$\begin{aligned}\mathcal{U}^P(\kappa, t) &= \mathbb{1}_{\{\kappa < k_*\}} \left(P(k_*) + \frac{P(k_*)}{k_*} (\kappa - k_*) \right) + \mathbb{1}_{\{k_* \leq \kappa \leq k^*\}} P(\kappa) + \mathbb{1}_{\{k^* < \kappa\}} (P(k^*) + \kappa - k^*), \\ \mathcal{L}^P(\kappa, t) &= \mathbb{1}_{\{z_*(t) < \kappa < k_*\}} (P(k_*) + \partial_{k+} P(k_*) (\kappa - k_*)) + \mathbb{1}_{\{k_* \leq \kappa \leq k^*\}} P(\kappa) \\ &\quad + \mathbb{1}_{\{k^* < \kappa < z^*(t)\}} (P(k^*) + \partial_{k-} P(k^*) (\kappa - k^*)) + \mathbb{1}_{\{z^*(t) \leq \kappa\}} (\kappa - 1).\end{aligned}$$

Obviously, the bounds $\mathcal{U}^\chi(\kappa, t) = \mathcal{L}^\chi(\kappa, t) = \chi(\kappa)$ at any interior strike $k_* \leq \kappa \leq k^*$ are immediately determined by the imposed constraints. However, as long as $P(k_*) > 0$ (with $z_*(t) > 0$) and $C(k^*) > 0$, it holds that $\mathcal{U}^\chi(\kappa, t) > \mathcal{L}^\chi(\kappa, t)$ for each $0 < \kappa < k_*$ in the left tail and for each $\kappa > k^*$ in the right tail, respectively. Here, the tail behavior of the bounds is entirely determined by the local shape of the observed option prices at the extreme strikes k_* and k^* , as characterized by the levels $P(k_*)$, $C(k^*)$ and slopes $\partial_{k+}P(k_*)$, $\partial_{k-}C(k^*)$.

Figure OE.2(a) illustrates the surfaces of option price bounds $\mathcal{L}^\chi(\kappa, \tau)$ and $\mathcal{U}^\chi(\kappa, \tau)$ in this case. In addition, Figure OE.3(a) plots the corresponding option price bounds $\mathcal{L}^\chi(\kappa, t)$ and $\mathcal{U}^\chi(\kappa, t)$ at the traded maturity t , which entirely determine these surfaces.

Tail probability bounds

For the tail probability bounds in Proposition 3.3, we likewise obtain simple, tractable expressions. We focus here on bounds at the traded maturity t . Unlike for option price bounds, the tail probability bounds at $\tau \neq t$ are not directly related to those at $\tau = t$, while still allowing for rather simple expressions. Without loss of generality, we suppose that $z_*(t) \leq k_*$ and $z^*(t) \geq k^*$ in order to simplify the exposition.

Under these conditions, the left-tail probability bounds in equations (3.22) and (3.23) evaluated at $\tau = t$ and any $\kappa \geq 0$ are given by

$$\begin{aligned}\mathcal{U}^{LTP}(\kappa, t) &= \mathbb{1}_{\{\kappa < z_*(t)\}} \frac{P(k_*)}{k_* - \kappa} + \mathbb{1}_{\{z_*(t) \leq \kappa < k^*\}} \partial_{k+}P(\kappa \vee k_*) + \mathbb{1}_{\{k^* \leq \kappa\}}, \\ \mathcal{L}^{LTP}(\kappa, t) &= \mathbb{1}_{\{z_*(t) \leq \kappa < k^*\}} \partial_{k+}P(\kappa \vee k_*) + \mathbb{1}_{\{k^* \leq \kappa < z^*(t)\}} \partial_{k-}P(k^*) + \mathbb{1}_{\{z^*(t) \leq \kappa\}} \frac{\kappa - 1 - P(k^*)}{\kappa - k^*}.\end{aligned}$$

The right-tail probability bounds in equations (3.24) and (3.25) evaluated at $\tau = t$ and any $\kappa \geq 0$ can be written as

$$\begin{aligned}\mathcal{U}^{RTP}(\kappa, t) &= \mathbb{1}_{\{\kappa \leq z_*(t)\}} + \mathbb{1}_{\{k_* < \kappa \leq z^*(t)\}} (-\partial_{k-}C(\kappa \wedge k^*)) + \mathbb{1}_{\{z^*(t) < \kappa\}} \frac{C(k^*)}{\kappa - k^*}, \\ \mathcal{L}^{RTP}(\kappa, t) &= \mathbb{1}_{\{\kappa \leq z_*(t)\}} \frac{1 - \kappa - C(k_*)}{k_* - \kappa} + \mathbb{1}_{\{z_*(t) < \kappa \leq k^*\}} (-\partial_{k+}C(k^*)).\end{aligned}$$

In case $z_*(t) = k_*$ or $z^*(t) = k^*$, the respective fractions in the lower bounds should be understood as directional limits.

Just as for option price bounds, the tail probability bounds at any interior strike are uniquely determined by the observed option prices. Concretely, we have that $\mathcal{U}^{LTP}(\kappa, t) = \mathcal{L}^{LTP}(\kappa, t) = \partial_{k+}P(\kappa)$ at any $k_* \leq \kappa < k^*$ as well as $\mathcal{U}^{RTP}(\kappa, t) = \mathcal{L}^{RTP}(\kappa, t) = -\partial_{k-}C(\kappa)$ at any $k_* < \kappa \leq k^*$. However, it holds that $\mathcal{U}^{XTP}(\kappa, t) > \mathcal{L}^{XTP}(\kappa, t)$ for each $0 < \kappa < k_*$ in the left tail and for each $\kappa > k^*$ in the right tail, provided that $P(k_*) > 0$ (with $z_*(t) > 0$) and $C(k^*) > 0$, respectively. The tail behavior of the bounds is again entirely determined by the levels $P(k_*)$, $C(k^*)$ and slopes $\partial_{k+}P(k_*)$, $\partial_{k-}C(k^*)$.

Figure OE.4(a) illustrates the tail probability bounds $\mathcal{L}^{XTP}(\kappa, t)$ and $\mathcal{U}^{XTP}(\kappa, t)$ in this case.

OE.2 Discrete strikes, single maturity, without bid-ask spreads

Consider the case without bid-ask spreads and a single maturity $\mathbb{T} := \{t\}$ with $t > 0$, where option prices are observed on a finite strike set $\mathbb{K} := \{k_1, \dots, k_n\} \subset (0, \bar{K})$ with $k_* := k_1 < 1 < k_n =: k^*$ and k_i sorted in strictly increasing order. In particular, suppose that $B^C(k, t) = A^C(k, t) = C(k)$ for all $k \in \mathbb{K}$,

where $C(k)$ is some function that represents arbitrage-free call prices in the sense of Lemma OA.2 and $P(k) = C(k) + k - 1$ yields the associated put prices.

Option price bounds

Analogous to the case with continuous strikes in section OE.1, it suffices to characterize the option price bounds in Proposition 3.1 at $\tau = t$. For these, we also obtain rather simple, tractable expressions in this case, equivalent to those stated in Theorem 4 of Bertsimas and Popescu (2002). Notably, the resulting bounds at any given κ depend only on observed option prices at immediately adjacent strikes in $\mathbb{K}$.

For $k_j < \kappa \leq k_{j+1}$, augmenting the strike grid by $k_0 = 0$ and $k_{n+1} = \infty$, the call price bounds in equations (3.6) and (3.7) evaluated at $\tau = t$ and any $\kappa \geq 0$ take the piecewise-linear form

$$\begin{aligned}\mathcal{U}^C(\kappa, t) &= \frac{k_{j+1} - \kappa}{k_{j+1} - k_j} C(k_j) + \frac{\kappa - k_j}{k_{j+1} - k_j} C(k_{j+1}), \\ \mathcal{L}^C(\kappa, t) &= \left(\frac{k_j - \kappa}{k_j - k_{j-1}} C(k_{j-1}) + \frac{\kappa - k_{j-1}}{k_j - k_{j-1}} C(k_j) \right) \vee \left(\frac{k_{j+2} - \kappa}{k_{j+2} - k_{j+1}} C(k_{j+1}) + \frac{\kappa - k_{j+1}}{k_{j+2} - k_{j+1}} C(k_{j+2}) \right),\end{aligned}$$

setting $C(k_{-1}) = 1 - k_{-1}$, $C(k_0) = 1$ and $C(k_{n+1}) = C(k_{n+2}) = 0$, while treating any term involving k_{n+1} (and k_{n+2}) as a limit. Analogously, for $k_j \leq \kappa < k_{j+1}$ with $k_0 = 0$ and $k_{n+1} = \infty$, the put price bounds in equations (3.8) and (3.9) at $\tau = t$ and any $\kappa \geq 0$ take the piecewise-linear form

$$\begin{aligned}\mathcal{U}^P(\kappa, t) &= \frac{k_{j+1} - \kappa}{k_{j+1} - k_j} P(k_j) + \frac{\kappa - k_j}{k_{j+1} - k_j} P(k_{j+1}), \\ \mathcal{L}^P(\kappa, t) &= \left(\frac{k_j - \kappa}{k_j - k_{j-1}} P(k_{j-1}) + \frac{\kappa - k_{j-1}}{k_j - k_{j-1}} P(k_j) \right) \vee \left(\frac{k_{j+2} - \kappa}{k_{j+2} - k_{j+1}} P(k_{j+1}) + \frac{\kappa - k_{j+1}}{k_{j+2} - k_{j+1}} P(k_{j+2}) \right),\end{aligned}$$

setting $P(k_{-1}) = P(k_0) = 0$ and $P(k_{n+1}) = k_{n+1} - k_n$, $P(k_{n+2}) = k_{n+2} - k_n$, while likewise treating any term involving k_{n+1} (and k_{n+2}) as a limit.

In this setting, option prices with maturity t are uniquely pinned down at any strike in $\mathbb{K}$, such that $\mathcal{U}^\chi(\kappa, t) = \mathcal{L}^\chi(\kappa, t) = \chi(\kappa)$ whenever $\kappa \in \mathbb{K}$. However, even within the strike range $[k_*, k^*]$, it generally holds that $\mathcal{U}^\chi(\kappa, t) > \mathcal{L}^\chi(\kappa, t)$ for any $\kappa \notin \mathbb{K}$. In particular, this relation holds in the left and right tails as long as $P(k_*) > 0$ (with $z_*(t) > 0$) and $C(k^*) > 0$, respectively.

Figure OE.2(b) illustrates the surfaces of option price bounds $\mathcal{L}^\chi(\kappa, \tau)$ and $\mathcal{U}^\chi(\kappa, \tau)$ in this case. In addition, Figure OE.3(b) plots the corresponding option price bounds $\mathcal{L}^\chi(\kappa, t)$ and $\mathcal{U}^\chi(\kappa, t)$ at the traded maturity t .

Tail probability bounds

Using the derived formulas for option price bounds in this case, we likewise obtain tractable expressions for the tail probability bounds in Proposition 3.3 at $\tau = t$, again depending only on observed option prices at immediately adjacent strikes in $\mathbb{K}$. In order to express the bounds in convenient form, define the left-hand finite differences $\Delta_{k-}C(k_i) := \frac{C(k_i) - C(k_{i-1})}{k_i - k_{i-1}}$ and $\Delta_{k-}P(k_i) := 1 + \Delta_{k-}C(k_i)$ as well as the right-hand finite differences $\Delta_{k+}C(k_i) := \frac{C(k_{i+1}) - C(k_i)}{k_{i+1} - k_i}$ and $\Delta_{k+}P(k_i) := 1 + \Delta_{k+}C(k_i)$.

For $k_j \leq \kappa < k_{j+1}$ with $k_0 = 0$ and $k_{n+1} = \infty$, the left-tail probability bounds in equations (3.22)

and (3.23) at $\tau = t$ and $\kappa \geq 0$ are given by

$$\begin{aligned}\mathcal{U}^{LTP}(\kappa, t) &= \left(\frac{k_{j+1} - k_j}{k_{j+1} - \kappa} \Delta_{k+} P(k_j) - \frac{\kappa - k_j}{k_{j+1} - \kappa} \Delta_{k-} P(k_j) \right) \wedge \Delta_{k+} P(k_{j+1}), \\ \mathcal{L}^{LTP}(\kappa, t) &= \Delta_{k-} P(k_j) \vee \left(\frac{k_{j+1} - k_j}{\kappa - k_j} \Delta_{k-} P(k_{j+1}) - \frac{k_{j+1} - \kappa}{\kappa - k_j} \Delta_{k+} P(k_{j+1}) \right),\end{aligned}$$

setting $\Delta_{k-} P(k_0) = 0$, $\Delta_{k+} P(k_0) = \frac{P(k_1)}{k_1}$ as well as $\Delta_{k-} P(k_{n+1}) = \Delta_{k+} P(k_{n+1}) = 1$. Similarly, for $k_j < \kappa \leq k_{j+1}$ with $k_0 = 0$ and $k_{n+1} = \infty$, the right-tail probability bounds in equations (3.24) and (3.25) at $\tau = t$ and any $\kappa \geq 0$ can be written as

$$\begin{aligned}\mathcal{U}^{RTP}(\kappa, t) &= -\Delta_{k-} C(k_j) \wedge \left(-\frac{k_{j+1} - k_j}{\kappa - k_j} \Delta_{k-} C(k_{j+1}) + \frac{k_{j+1} - \kappa}{\kappa - k_j} \Delta_{k+} C(k_{j+1}) \right), \\ \mathcal{L}^{RTP}(\kappa, t) &= \left(-\frac{k_{j+1} - k_j}{k_{j+1} - \kappa} \Delta_{k+} C(k_j) + \frac{\kappa - k_j}{k_{j+1} - \kappa} \Delta_{k-} C(k_j) \right) \vee -\Delta_{k+} C(k_{j+1}),\end{aligned}$$

setting $\Delta_{k-} C(k_0) = -1$, $\Delta_{k+} C(k_0) = \frac{C(k_1) - 1}{k_1}$ as well as $\Delta_{k-} C(k_{n+1}) = \Delta_{k+} C(k_{n+1}) = 0$.

In general, it now holds that $\mathcal{U}^{XTP}(\kappa, t) > \mathcal{L}^{XTP}(\kappa, t)$ for all $\kappa > 0$, even if κ corresponds to a traded strike in $\mathbb{K}$ at which the option price is uniquely determined. Concretely, if $\kappa = k_j \in \mathbb{K}$, we have $\mathcal{U}^{LTP}(\kappa, t) = \Delta_{k+} P(k_j) \geq \Delta_{k-} P(k_j) = \mathcal{L}^{LTP}(\kappa, t)$ (unless $\Delta_{k-} P(k_{j+1}) = \Delta_{k+} P(k_{j+1})$) as well as $\mathcal{U}^{RTP}(\kappa, t) = -\Delta_{k-} C(k_j) \geq -\Delta_{k+} C(k_j) = \mathcal{L}^{RTP}(\kappa, t)$ (unless $\Delta_{k-} C(k_{j-1}) = \Delta_{k+} C(k_{j-1})$), where equality only holds in boundary cases.

Figure OE.4(b) illustrates the tail probability bounds $\mathcal{L}^{XTP}(\kappa, t)$ and $\mathcal{U}^{XTP}(\kappa, t)$ in this case.

OE.3 Discrete strikes, single maturity, with bid-ask spreads

We now extend the setting in section OE.2 to non-zero bid-ask spreads. Specifically, still consider a single maturity $\mathbb{T} := \{t\}$ with $t > 0$ and a finite strike set $\mathbb{K} := \{k_1, \dots, k_n\} \subset (0, \bar{K})$ with $k_* := k_1 < 1 < k_n =: k^*$ and strictly increasing k_i . Bid and ask prices are observed for strikes in $\mathbb{K}$ such that $(1 - k)^+ \leq B^C(k, t) < C(k) < A^C(k, t) \leq 1$ for all $k \in \mathbb{K}$, where $C(k)$ is some function that represents arbitrage-free call prices in the sense of Lemma OA.2.

Option price bounds

As before, it suffices to characterize the option price bounds in Proposition 3.1 at the traded maturity t . Without further structure on the bid-ask prices, the resulting expressions do not simplify considerably.

Generally in this case, the call price bounds in equations (3.6) and (3.7) at $\tau = t$ and any $\kappa \geq 0$ take the piecewise-linear form

$$\begin{aligned}\mathcal{U}^C(\kappa, t) &= 1 \wedge \min \{ A^C(k', t) : k' \leq \kappa \} \wedge \min \left\{ \frac{k' - \kappa}{k'} + \frac{\kappa}{k'} A^C(k', t) : 0 < \kappa < k' \right\} \\ &\quad \wedge \min \left\{ \frac{k'' - \kappa}{k'' - k'} A^C(k', t) + \frac{\kappa - k'}{k'' - k'} A^C(k'', t) : k' \leq \kappa < k'' \right\},\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^C(\kappa, t) = & (1 - \kappa)^+ \vee \max\{B^C(k', t) : \kappa \leq k'\} \vee \max\left\{\frac{k' - \kappa}{k'} + \frac{\kappa}{k'} B^C(k', t) : 0 < k' \leq \kappa\right\} \\
& \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} B^C(k', t) + \frac{\kappa - k'}{k'' - k'} A^C(k'', t) : \kappa \leq k' < k''\right\} \\
& \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} A^C(k', t) + \frac{\kappa - k'}{k'' - k'} B^C(k'', t) : k' < k'' \leq \kappa\right\}.
\end{aligned}$$

Analogously, the put price bounds in equations (3.8) and (3.9) at $\tau = t$ and any $\kappa \geq 0$ take the piecewise-linear form

$$\begin{aligned}
\mathcal{U}^P(\kappa, t) = & \kappa \wedge \min\{A^P(k', t) + \kappa - k' : k' \leq \kappa\} \wedge \min\left\{\frac{\kappa}{k'} A^P(k', t) : 0 < \kappa < k'\right\} \\
& \wedge \min\left\{\frac{k'' - \kappa}{k'' - k'} A^P(k', t) + \frac{\kappa - k'}{k'' - k'} A^P(k'', t) : k' \leq \kappa < k''\right\}, \\
\mathcal{L}^P(\kappa, t) = & (\kappa - 1)^+ \vee \max\{B^P(k', t) + \kappa - k' : \kappa \leq k'\} \vee \max\left\{\frac{\kappa}{k'} B^P(k', t) : 0 < k' \leq \kappa\right\} \\
& \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} B^P(k', t) + \frac{\kappa - k'}{k'' - k'} A^P(k'', t) : \kappa \leq k' < k''\right\} \\
& \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} A^P(k', t) + \frac{\kappa - k'}{k'' - k'} B^P(k'', t) : k' < k'' \leq \kappa\right\}.
\end{aligned}$$

To more easily compare to the bounds in the case without bid-ask spreads, we may rewrite the option price bounds in terms of the optimal strikes $k_{I_*} \leq \kappa < k_{I^*}$ as well as $k_{J_*} < k_{J^*} \leq \kappa$ and $\kappa \leq k_{J^{**}} < k_{J^{**}}$ that realize the respective minima and maxima for a given κ , augmenting the strike set by $k_0 = 0$ and $k_{n+1} = \infty$ with $A^C(k_0, t) = 1$ and $A^C(k_{n+1}, t) = 0$. In terms of these indices, which depend implicitly on κ , the call price bounds in equations (3.6) and (3.7) at $\tau = t$ and $k_* \leq \kappa \leq k^*$ may be expressed as

$$\begin{aligned}
\mathcal{U}^C(\kappa, t) = & \frac{k_{I^*} - \kappa}{k_{I^*} - k_{I_*}} A^C(k_{I_*}, t) + \frac{\kappa - k_{I_*}}{k_{I^*} - k_{I_*}} A^C(k_{I^*}, t), \\
\mathcal{L}^C(\kappa, t) = & \left(\frac{k_{J^*} - \kappa}{k_{J^*} - k_{J_*}} A^C(k_{J_*}, t) + \frac{\kappa - k_{J_*}}{k_{J^*} - k_{J_*}} B^C(k_{J^*}, t) \right) \\
& \vee \left(\frac{k_{J^{**}} - \kappa}{k_{J^{**}} - k_{J^{**}}} B^C(k_{J^{**}}, t) + \frac{\kappa - k_{J^{**}}}{k_{J^{**}} - k_{J^{**}}} A^C(k_{J^{**}}, t) \right).
\end{aligned}$$

Analogous expressions obtain also for the corresponding put price bounds in equations (3.8) and (3.9).

Figure OE.2(c) illustrates the surfaces of option price bounds $\mathcal{L}^X(\kappa, \tau)$ and $\mathcal{U}^X(\kappa, \tau)$ in this case. In addition, Figure OE.3(c) plots the corresponding option price bounds $\mathcal{L}^X(\kappa, t)$ and $\mathcal{U}^X(\kappa, t)$ at the traded maturity t .

Notably, the bounds resulting from our Proposition 3.1 in this case differ substantially from the expressions stated in Theorem 7 of Bertsimas and Popescu (2002) within an equivalent setting. As for the case without bid-ask spreads, the latter again rely only on immediately adjacent strikes. Specifically, supposing that $k_j < \kappa < k_{j+1}$, they differ as follows. For the upper bound $\mathcal{U}^C(\kappa, t)$, they use $I_* = j$, $I^* = j + 1$. This generally corresponds to the upper bound in equation (3.6) if and only if ask prices are arbitrage-free from the outset, which is not guaranteed. If this is not the case, then the alleged upper bound is too loose relative to equation (3.6). For the lower bound $\mathcal{L}^C(\kappa, t)$, they use $J_* = j - 1$, $J^* = j$ and $J^{**} = j + 1$, $J^{**} = j + 2$, while at the same time employing bid prices $B^C(k_{J_*}, t)$, $B^C(k_{J^{**}}, t)$ instead of ask prices. This generally cannot be reconciled with equation (3.7) as ask prices do not play a role in their expression. As a result, the alleged lower bound can be either too tight or too loose relative to equation (3.7), depending on the specific structure of the observed bid-ask prices.

Figure OE.1 provides an illustrative comparison of the option price bounds in Proposition 3.1 and Theorem 7 of Bertsimas and Popescu (2002).

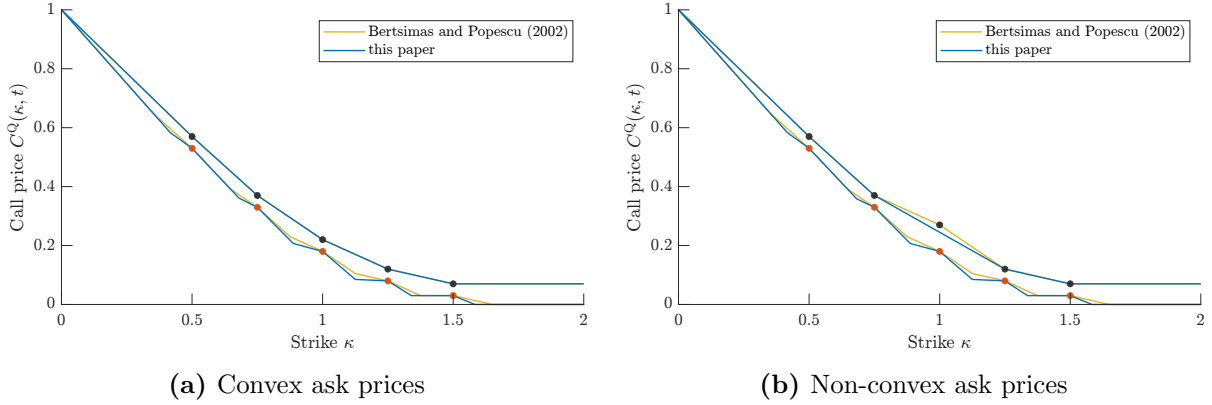


Figure OE.1: Comparison with Bertsimas and Popescu (2002) bounds

This figure compares the option price bounds in Proposition 3.1 to the bounds in Theorem 7 of Bertsimas and Popescu (2002). Option prices are observed on a strike grid $\mathbb{K} = \{0.5, 0.75, 1, 1.25, 1.5\}$ for the single maturity $\mathbb{T} = \{t\}$. Mid prices are specified as $C(0.5) = 0.55$, $C(0.75) = 0.35$, $C(1) = 0.2$, $C(1.25) = 0.1$, $C(1.5) = 0.05$. Bid and ask option prices are determined as $B^C(k, t) = C(k) - s/2$ and $A^C(k, t) = C(k) + s/2$ for all $k \in \mathbb{K}$ given a bid-ask spread $s = 0.04$. In panel (a), we use $B^C(k, t)$ and $A^C(k, t)$; in panel (b), we use $B^C(k, t)$ and $A^C(k, t)$ except for $A^C(1, t) = C(1) + 0.07$. The colored lines plot the option price bounds $\mathcal{L}^C(k, t)$ and $\mathcal{U}^C(k, t)$.

Tail probability bounds

With a finite strike set $\mathbb{K}$, the lower bound $\mathcal{L}^{LTP}(k, \tau)$ in equation (3.23) can be expressed equivalently as

$$\mathcal{L}^{LTP}(k, \tau) = \left(\partial_{k+} \mathcal{L}^P(k, \tau) + \lim_{\eta \downarrow 0} \frac{\mathcal{L}^P(k, \tau) - \mathcal{U}^P(k, \tau)}{\eta} \right) \vee \max \left\{ \frac{\mathcal{L}^P(k, \tau) - \mathcal{U}^P(k', \tau)}{k - k'} : k' < k \right\},$$

where the term in parentheses equals $\partial_{k+} \mathcal{L}^P(k, \tau)$ if $\mathcal{L}^P(k, \tau) = \mathcal{U}^P(k, \tau)$ and otherwise $-\infty$. Similarly, the lower bound $\mathcal{L}^{RTP}(k, \tau)$ in equation (3.25) can be expressed equivalently as

$$\mathcal{L}^{RTP}(k, \tau) = 0 \vee \left(-\partial_{k-} \mathcal{L}^C(k, \tau) - \lim_{\eta \downarrow 0} \frac{\mathcal{U}^C(k, \tau) - \mathcal{L}^C(k, \tau)}{\eta} \right) \vee \max \left\{ -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(k, \tau)}{k' - k} : k < k' \right\},$$

where the term in parentheses equals $-\partial_{k-} \mathcal{L}^C(k, \tau)$ if $\mathcal{L}^C(k, \tau) = \mathcal{U}^C(k, \tau)$ and otherwise $-\infty$.

In the current setting, the bounds for left-tail probabilities in equations (3.22) and (3.23) at $\tau = t$ and $k \geq 0$ can thus be written as

$$\begin{aligned} \mathcal{U}^{LTP}(k, t) &= 1 \wedge \min \left\{ \frac{\mathcal{U}^P(k', t) - \mathcal{L}^P(k, t)}{k' - k} : k < k' \right\}, \\ \mathcal{L}^{LTP}(k, t) &= \max \left\{ \frac{\mathcal{L}^P(k, t) - \mathcal{U}^P(k', t)}{k - k'} : k' < k \right\}. \end{aligned}$$

Likewise, the bounds for right-tail probabilities in equations (3.24) and (3.25) at $\tau = t$ and $k \geq 0$ can be

given as

$$\begin{aligned}\mathcal{U}^{RTP}(\kappa, t) &= \min \left\{ -\frac{\mathcal{L}^C(\kappa, t) - \mathcal{U}^C(k', t)}{\kappa - k'} : k' < \kappa \right\}, \\ \mathcal{L}^{RTP}(\kappa, t) &= 0 \vee \max \left\{ -\frac{\mathcal{U}^C(k', t) - \mathcal{L}^C(\kappa, t)}{k' - \kappa} : \kappa < k' \right\}.\end{aligned}$$

It follows that there exist optimal strikes $k_{I*} < \kappa < k_{I*}$ and $k_{J*} < \kappa < k_{J*}$, depending implicitly on κ , that realize the respective minima and maxima. As usual, we may augment the strike set by $k_0 = 0$ and $k_{n+1} = \infty$ when considering appropriate limits. In terms of these optimal strikes, we can then write the bounds for left-tail probabilities in equations (3.22) and (3.23) at $\tau = t$ and $\kappa \geq 0$ as

$$\begin{aligned}\mathcal{U}^{LTP}(\kappa, t) &= \frac{\mathcal{U}^P(k_{I*}, t) - \mathcal{L}^P(\kappa, t)}{k_{I*} - \kappa}, \\ \mathcal{L}^{LTP}(\kappa, t) &= \frac{\mathcal{L}^P(\kappa, t) - \mathcal{U}^P(k_{I*}, t)}{\kappa - k_{I*}},\end{aligned}$$

and the bounds for right-tail probabilities in equations (3.24) and (3.25) at $\tau = t$ and $\kappa \geq 0$ as

$$\begin{aligned}\mathcal{U}^{RTP}(\kappa, t) &= -\frac{\mathcal{L}^C(\kappa, t) - \mathcal{U}^C(k_{J*}, t)}{\kappa - k_{J*}}, \\ \mathcal{L}^{RTP}(\kappa, t) &= -\frac{\mathcal{U}^C(k_{J*}, t) - \mathcal{L}^C(\kappa, t)}{k_{J*} - \kappa}.\end{aligned}$$

As an immediate consequence, we have that $\mathcal{U}^{XTP}(\kappa, t) > \mathcal{L}^{XTP}(\kappa, t)$ for all $\kappa > 0$.

Figure OE.4(c) illustrates the tail probability bounds $\mathcal{L}^{XTP}(\kappa, t)$ and $\mathcal{U}^{XTP}(\kappa, t)$ in this case.

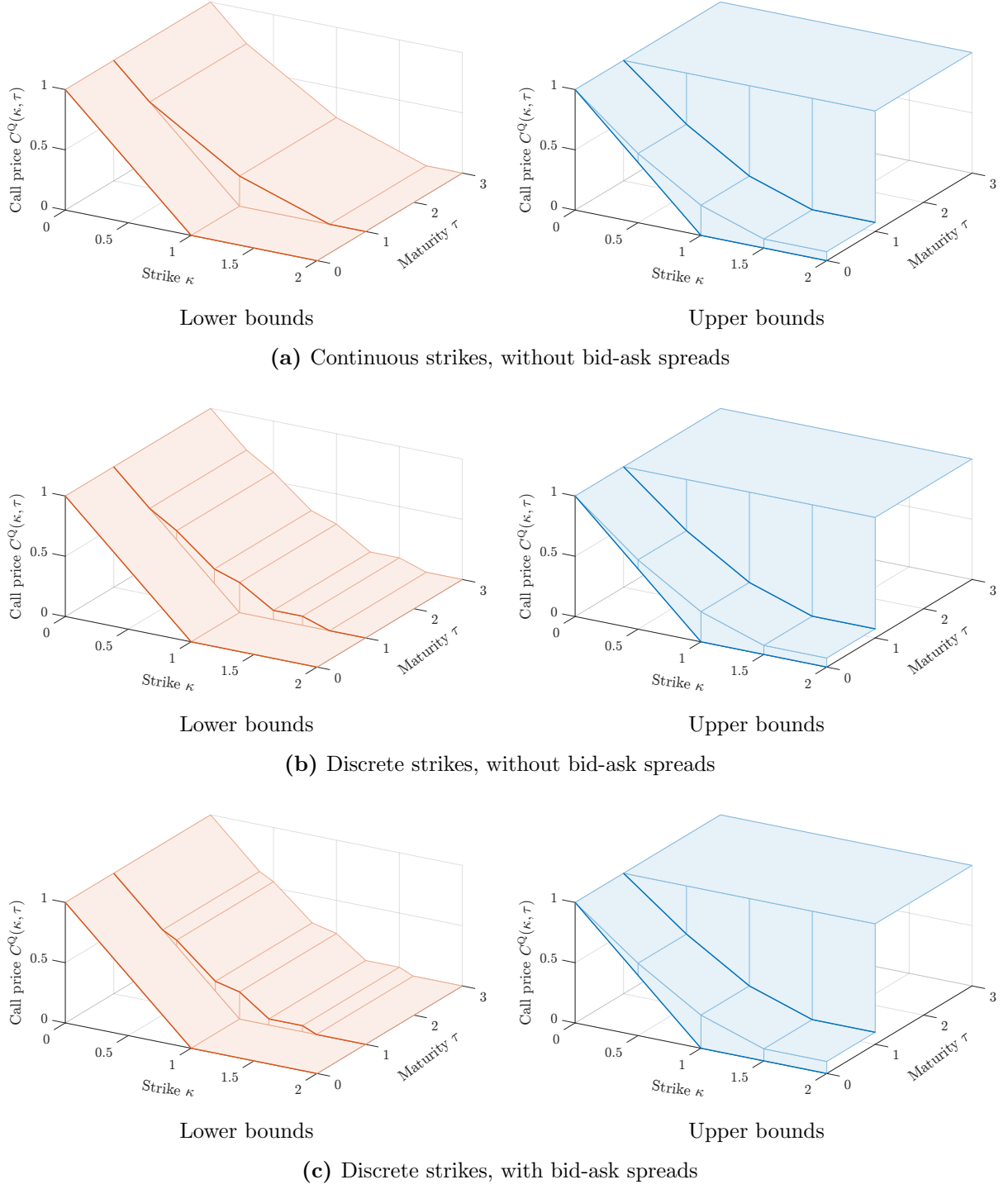
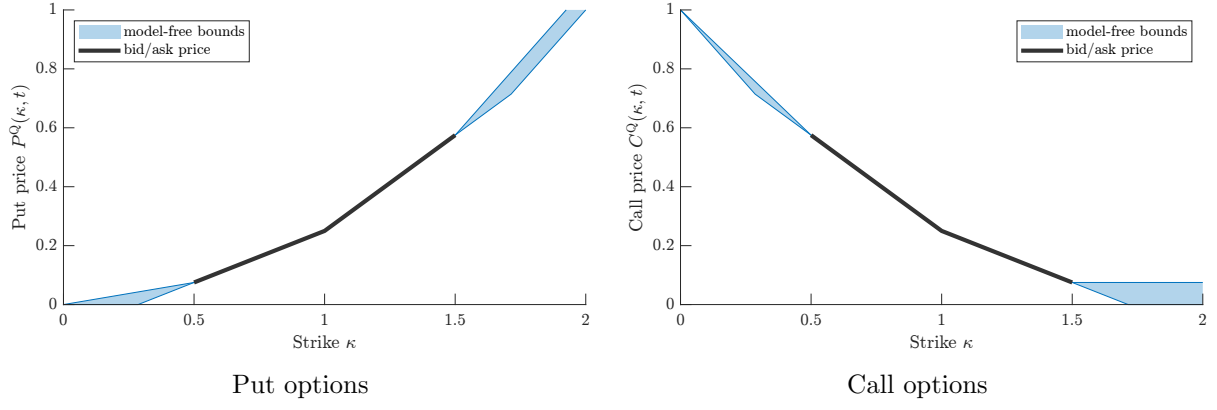
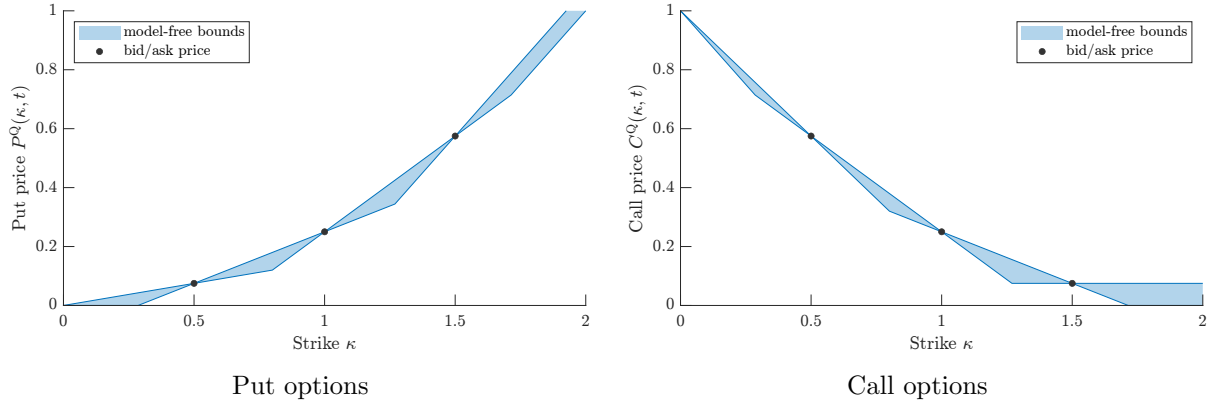


Figure OE.2: Illustrative option price bounds, single maturity

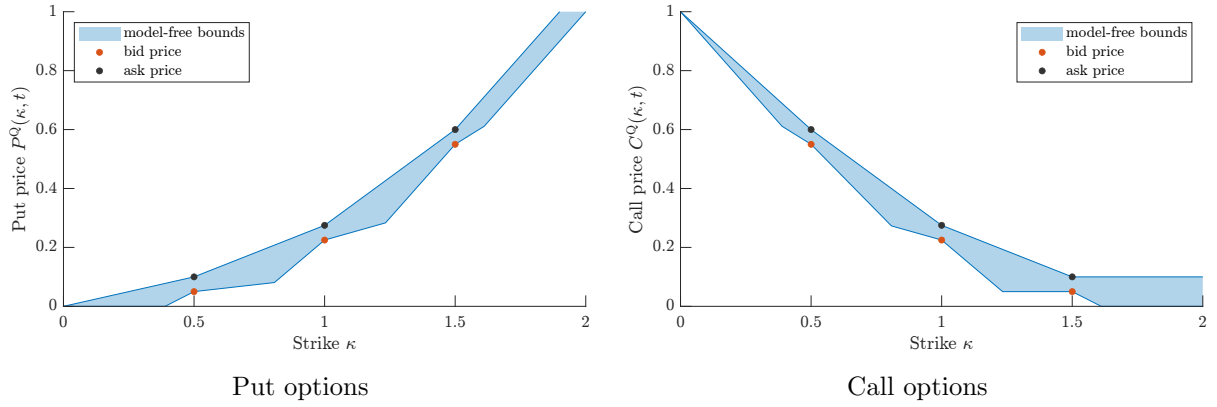
This figure illustrates the option price bounds in Proposition 3.1 within the settings of sections OE.1 to OE.3. Option prices are observed for the single maturity $\mathbb{T} = \{1\}$. Mid prices $C(k)$ are specified as $C(0.5) = 0.575$, $C(1) = 0.25$, $C(1.5) = 0.075$, otherwise using linear interpolation. Bid and ask option prices are determined as $B^C(k, t) = C(k) - s/2$ and $A^C(k, t) = C(k) + s/2$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ given a bid-ask spread s . In panel (a), we set $\mathbb{K} = [0.5, 1.5]$ and $s = 0$; in panel (b), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0$; in panel (c), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0.05$. In each panel, the left and right plots show the lower bounds $\mathcal{L}^C(k, \tau)$ and upper bounds $\mathcal{U}^C(k, \tau)$, respectively.



(a) Continuous strikes, without bid-ask spreads



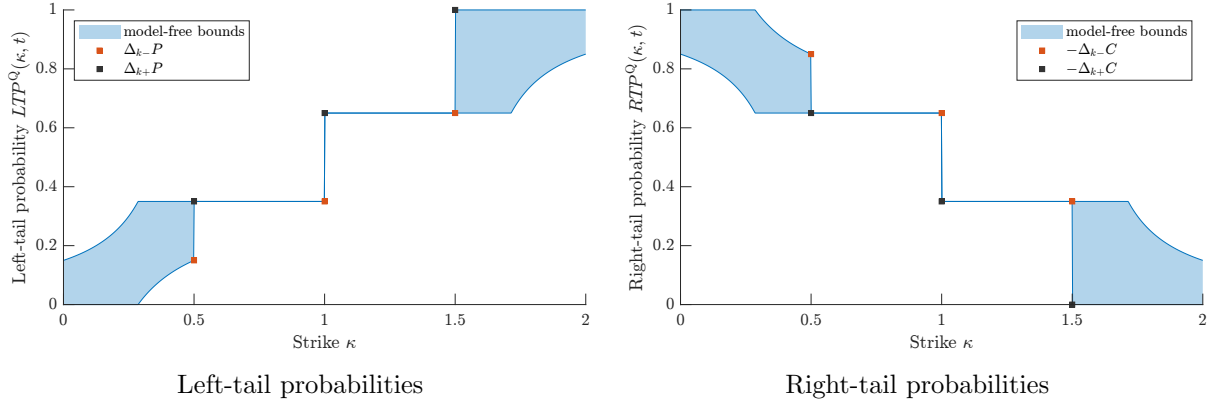
(b) Discrete strikes, without bid-ask spreads



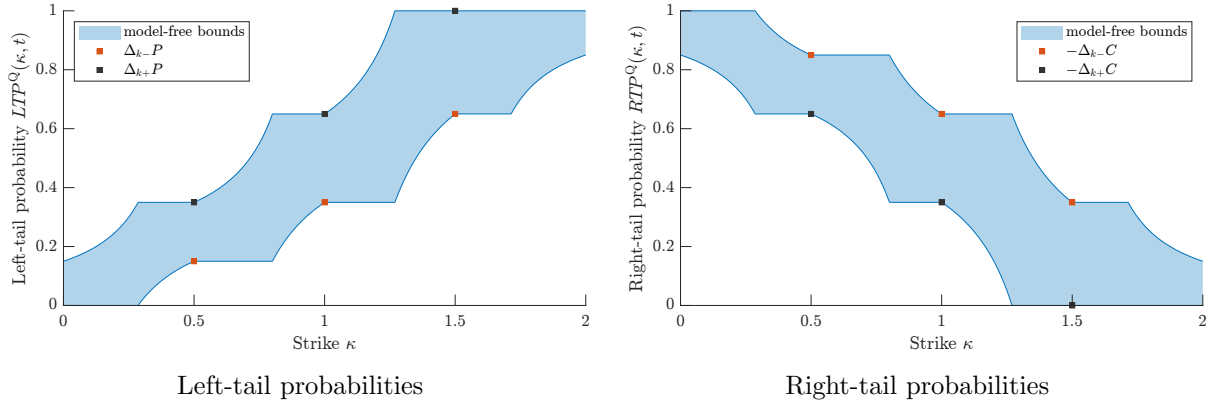
(c) Discrete strikes, with bid-ask spreads

Figure OE.3: Illustrative option price bounds, single maturity

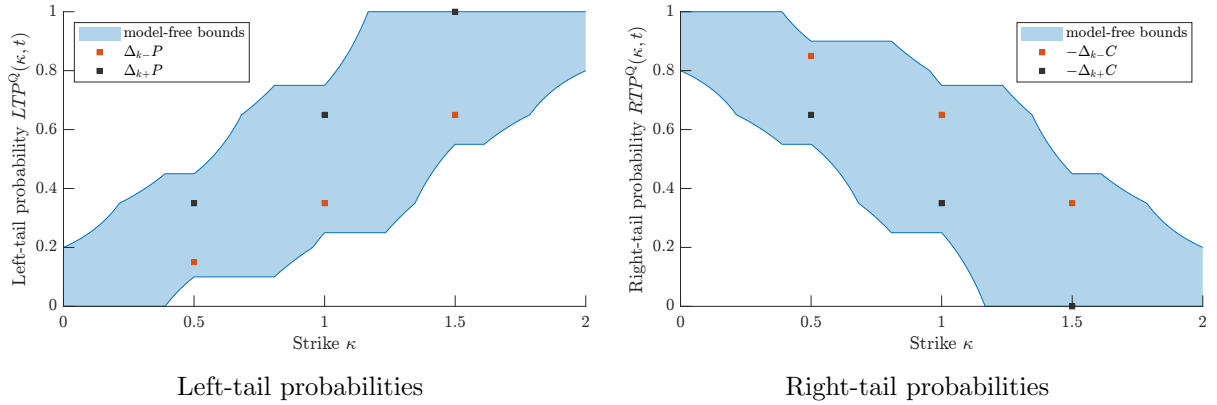
This figure illustrates the option price bounds in Proposition 3.1 within the settings of sections OE.1 to OE.3. Option prices are observed for the single maturity $\mathbb{T} = \{1\}$. Mid prices $C(k)$ are specified as $C(0.5) = 0.575$, $C(1) = 0.25$, $C(1.5) = 0.075$, otherwise using linear interpolation. Bid and ask option prices are determined as $B^C(k, t) = C(k) - s/2$ and $A^C(k, t) = C(k) + s/2$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ given a bid-ask spread s . In panel (a), we set $\mathbb{K} = [0.5, 1.5]$ and $s = 0$; in panel (b), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0$; in panel (c), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0.05$. The blue lines and shaded areas plot the option price bounds $\mathcal{L}^X(k, t)$ and $\mathcal{U}^X(k, t)$ for $t = 1$.



(a) Continuous strikes, without bid-ask spreads



(b) Discrete strikes, without bid-ask spreads



(c) Discrete strikes, with bid-ask spreads

Figure OE.4: Illustrative tail probability bounds, single maturity

This figure illustrates the tail probability bounds in Proposition 3.3 within the settings of sections OE.1 to OE.3. Option prices are observed for the single maturity $\mathbb{T} = \{1\}$. Mid prices $C(k)$ are specified as $C(0.5) = 0.575$, $C(1) = 0.25$, $C(1.5) = 0.075$, otherwise using linear interpolation. Bid and ask option prices are determined as $B^C(k, t) = C(k) - s/2$ and $A^C(k, t) = C(k) + s/2$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ given a bid-ask spread s . In panel (a), we set $\mathbb{K} = [0.5, 1.5]$ and $s = 0$; in panel (b), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0$; in panel (c), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0.05$. The blue lines and shaded areas plot the tail probability bounds $\mathcal{L}^{XTP}(\kappa, t)$ and $\mathcal{U}^{XTP}(\kappa, t)$ for $t = 1$.

OE.4 Continuous strikes, multiple maturities, without bid-ask spreads

As an extension of the single-maturity case in section OE.1, consider a setting where option prices without bid-ask spreads are observed at multiple maturities $\mathbb{T} := \{t_1, \dots, t_m\}$ with $t_j > 0$ and sorted in strictly increasing order. At each traded maturity in $\mathbb{T}$, option prices are observed for every strike in the interval $\mathbb{K} := [k_*, k^*] \subset (0, \bar{K})$ with $k_* < 1 < k^*$. Bid and ask prices are given by $B^C(k, t) = A^C(k, t) = C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, such that $C(k, t)$ represents an arbitrage-free call price surface, satisfying the conditions in Lemma OA.2, while $P(k, t) = C(k, t) + k - 1$ yields the associated put price surface.

Option price bounds

To obtain tractable expressions for the option price bounds in Proposition 3.1 in the present setting, note that Corollary OC.3 establishes a simple relation to the single-period bounds derived in section OE.1. In this regard, note that the conditions of Corollary OC.3 are satisfied by construction.

Hence, we define $I^* := \min\{i : t_i \geq \tau\}$ and $I_* := \max\{i : t_i \leq \tau\}$ given a particular $\tau \geq 0$, using $t_0 = 0$ and $t_{m+1} = \infty$ with $C(k, t_0) = (1 - k)^+$ and $C(k, t_{m+1}) = 1$. The call price bounds in equations (3.6) and (3.7) for $\tau \geq 0$ and $\kappa \geq 0$ then take the form

$$\begin{aligned} \mathcal{U}^C(\kappa, \tau) &= C((\kappa \vee k_*) \wedge k^*, t_{I^*}) - \frac{C(k_*, t_{I^*}) - 1}{k_*} (k_* - \kappa)^+, \\ \mathcal{L}^C(\kappa, \tau) &= (1 - \kappa)^+ \vee \bigvee_{i \leq I_*} \left(C((\kappa \vee k_*) \wedge k^*, t_i) - \partial_{k+} C(k_*, t_i) (k_* - \kappa)^+ + \partial_{k-} C(k^*, t_i) (\kappa - k^*)^+ \right). \end{aligned}$$

Analogously, the put price bounds in equations (3.8) and (3.9) for $\tau \geq 0$ and $\kappa \geq 0$ can be expressed as

$$\begin{aligned} \mathcal{U}^P(\kappa, \tau) &= P((\kappa \vee k_*) \wedge k^*, t_{I^*}) - \frac{P(k_*, t_{I^*})}{k_*} (k_* - \kappa)^+ + (\kappa - k^*)^+, \\ \mathcal{L}^P(\kappa, \tau) &= (\kappa - 1)^+ \vee \bigvee_{i \leq I_*} \left(P((\kappa \vee k_*) \wedge k^*, t_i) - \partial_{k+} P(k_*, t_i) (k_* - \kappa)^+ + \partial_{k-} P(k^*, t_i) (\kappa - k^*)^+ \right). \end{aligned}$$

Figure OE.5(a) illustrates the surfaces of option price bounds $\mathcal{L}^X(\kappa, \tau)$ and $\mathcal{U}^X(\kappa, \tau)$ in this case.

OE.5 Discrete strikes, multiple maturities, without bid-ask spreads

As an extension of the single-maturity case in section OE.2, now take multiple maturities $\mathbb{T} := \{t_1, \dots, t_m\}$ with strictly increasing $t_j > 0$ as well as a finite strike set $\mathbb{K} := \{k_1, \dots, k_n\} \subset (0, \bar{K})$ with $k_* := k_1 < 1 < k_n =: k^*$ and strictly increasing k_i . Bid and ask prices are given by $B^C(k, t) = A^C(k, t) = C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, where $C(k, t)$ is some arbitrage-free call price surface in the sense of Lemma OA.2 and $P(k, t) = C(k, t) + k - 1$ is the associated put price surface.

Option price bounds

Similar to the continuous case in section OE.4, we likewise invoke Corollary OC.3 to obtain tractable expressions for the option price bounds in Proposition 3.1, exploiting the single-period bounds derived in section OE.2. Also in the current setting, the conditions of Corollary OC.3 are satisfied by construction.

Therefore, we again define $I^* := \min\{i : t_i \geq \tau\}$ and $I_* := \max\{i : t_i \leq \tau\}$ for a given $\tau \geq 0$, using $t_0 = 0$ and $t_{m+1} = \infty$ with $C(k, 0) = (1 - k)^+$ and $C(k, \infty) = 1$. The call price bounds in equations (3.6)

and (3.7) for $\tau \geq 0$ and $k_j < \kappa \leq k_{j+1}$ can be written as

$$\begin{aligned}\mathcal{U}^C(\kappa, \tau) &= \frac{k_{j+1} - \kappa}{k_{j+1} - k_j} C(k_j, t_{I^*}) + \frac{\kappa - k_j}{k_{j+1} - k_j} C(k_{j+1}, t_{I^*}), \\ \mathcal{L}^C(\kappa, \tau) &= \bigvee_{i \leq I_*} \left(\frac{k_j - \kappa}{k_j - k_{j-1}} C(k_{j-1}, t_i) + \frac{\kappa - k_{j-1}}{k_j - k_{j-1}} C(k_j, t_i) \right) \\ &\quad \vee \left(\frac{k_{j+2} - \kappa}{k_{j+2} - k_{j+1}} C(k_{j+1}, t_i) + \frac{\kappa - k_{j+1}}{k_{j+2} - k_{j+1}} C(k_{j+2}, t_i) \right).\end{aligned}$$

The put price bounds in equations (3.8) and (3.9) for $\tau \geq 0$ and $k_j \leq \kappa < k_{j+1}$ are given by

$$\begin{aligned}\mathcal{U}^P(\kappa, \tau) &= \frac{k_{j+1} - \kappa}{k_{j+1} - k_j} P(k_j, t_{I^*}) + \frac{\kappa - k_j}{k_{j+1} - k_j} P(k_{j+1}, t_{I^*}), \\ \mathcal{L}^P(\kappa, \tau) &= \bigvee_{i \leq I_*} \left(\frac{k_j - \kappa}{k_j - k_{j-1}} P(k_{j-1}, t_i) + \frac{\kappa - k_{j-1}}{k_j - k_{j-1}} P(k_j, t_i) \right) \\ &\quad \vee \left(\frac{k_{j+2} - \kappa}{k_{j+2} - k_{j+1}} P(k_{j+1}, t_i) + \frac{\kappa - k_{j+1}}{k_{j+2} - k_{j+1}} P(k_{j+2}, t_i) \right).\end{aligned}$$

Figure OE.5(b) illustrates the surfaces of option price bounds $\mathcal{L}^X(\kappa, \tau)$ and $\mathcal{U}^X(\kappa, \tau)$ in this case.

OE.6 Discrete strikes, multiple maturities, with bid-ask spreads

Analogous to the single-maturity case in section OE.3, we now augment the multi-maturity setting in section OE.5 with non-zero bid-ask spreads. Thus, we again consider multiple maturities $\mathbb{T} := \{t_1, \dots, t_m\}$ with strictly increasing $t_j > 0$ as well as a finite strike set $\mathbb{K} := \{k_1, \dots, k_n\} \subset (0, \bar{K})$ with $k_* := k_1 < 1 < k_n =: k^*$ and strictly increasing k_i . However, bid and ask prices satisfy $(1 - k)^+ \leq B^C(k, t) < C(k, t) < A^C(k, t) \leq 1$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$, where $C(k, t)$ is an arbitrage-free call price surface in the sense of Lemma OA.2.

Option price bounds

In order to invoke Corollary OC.3 in this setting to simplify the formulas for option price bounds in Proposition 3.1, we need a mild assumption. Specifically, we additionally suppose that $A^C(k, t_i) \leq A^C(k, t_{i+1})$ for all $k \in \mathbb{K}$.

This additional assumption implies the conditions in Corollary OC.3 and thus allows to exploit the expressions derived in section OE.3. To do so, we again employ the definitions $I^* := \min\{i : t_i \geq \tau\}$ and $I_* := \max\{i : t_i \leq \tau\}$ for a given $\tau \geq 0$, using $t_0 = 0$ and $t_{m+1} = \infty$ with $C(k, 0) = (1 - k)^+$ and $C(k, \infty) = 1$. Thereby, the call price bounds in equations (3.6) and (3.7) with $\tau \geq 0$ and $\kappa \geq 0$ can be expressed as

$$\begin{aligned}\mathcal{U}^C(\kappa, \tau) &= 1 \wedge \min\{A^C(k', t_{I^*}) : k' \leq \kappa\} \wedge \min\left\{\frac{k' - \kappa}{k'} + \frac{\kappa}{k'} A^C(k', t_{I^*}) : 0 < \kappa < k'\right\} \\ &\quad \wedge \min\left\{\frac{k'' - \kappa}{k'' - k'} A^C(k', t_{I^*}) + \frac{\kappa - k'}{k'' - k'} A^C(k'', t_{I^*}) : k' \leq \kappa < k''\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{L}^C(\kappa, \tau) = (1 - \kappa)^+ \vee \bigvee_{i \leq I_*} & \left(\max\{B^C(k', t_i) : \kappa \leq k'\} \vee \max\left\{\frac{k' - \kappa}{k'} + \frac{\kappa}{k'} B^C(k', t_i) : 0 < k' \leq \kappa\right\} \right. \\ & \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} B^C(k', t_i) + \frac{\kappa - k'}{k'' - k'} A^C(k'', t_i) : \kappa \leq k' < k''\right\} \\ & \left. \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} A^C(k', t_i) + \frac{\kappa - k'}{k'' - k'} B^C(k'', t_i) : k' < k'' \leq \kappa\right\} \right).\end{aligned}$$

Moreover, the put price bounds in equations (3.8) and (3.9) with $\tau \geq 0$ and $\kappa \geq 0$ can be written as

$$\begin{aligned}\mathcal{U}^P(\kappa, \tau) = \kappa \wedge \min\{A^P(k', t_{I^*}) + \kappa - k' : k' \leq \kappa\} \wedge \min\left\{\frac{\kappa}{k'} A^P(k', t_{I^*}) : 0 < \kappa < k'\right\} \\ \wedge \min\left\{\frac{k'' - \kappa}{k'' - k'} A^P(k', t_{I^*}) + \frac{\kappa - k'}{k'' - k'} A^P(k'', t_{I^*}) : k' \leq \kappa < k''\right\}, \\ \mathcal{L}^P(\kappa, \tau) = (\kappa - 1)^+ \vee \bigvee_{i \leq I_*} & \left(\max\{B^P(k', t_i) + \kappa - k' : \kappa \leq k'\} \vee \max\left\{\frac{\kappa}{k'} B^P(k', t_i) : 0 < k' \leq \kappa\right\} \right. \\ & \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} B^P(k', t_i) + \frac{\kappa - k'}{k'' - k'} A^P(k'', t_i) : \kappa \leq k' < k''\right\} \\ & \left. \vee \max\left\{\frac{k'' - \kappa}{k'' - k'} A^P(k', t_i) + \frac{\kappa - k'}{k'' - k'} B^P(k'', t_i) : k' < k'' \leq \kappa\right\} \right).\end{aligned}$$

Figure OE.5(c) illustrates the surfaces of option price bounds $\mathcal{L}^X(\kappa, \tau)$ and $\mathcal{U}^X(\kappa, \tau)$ in this case.

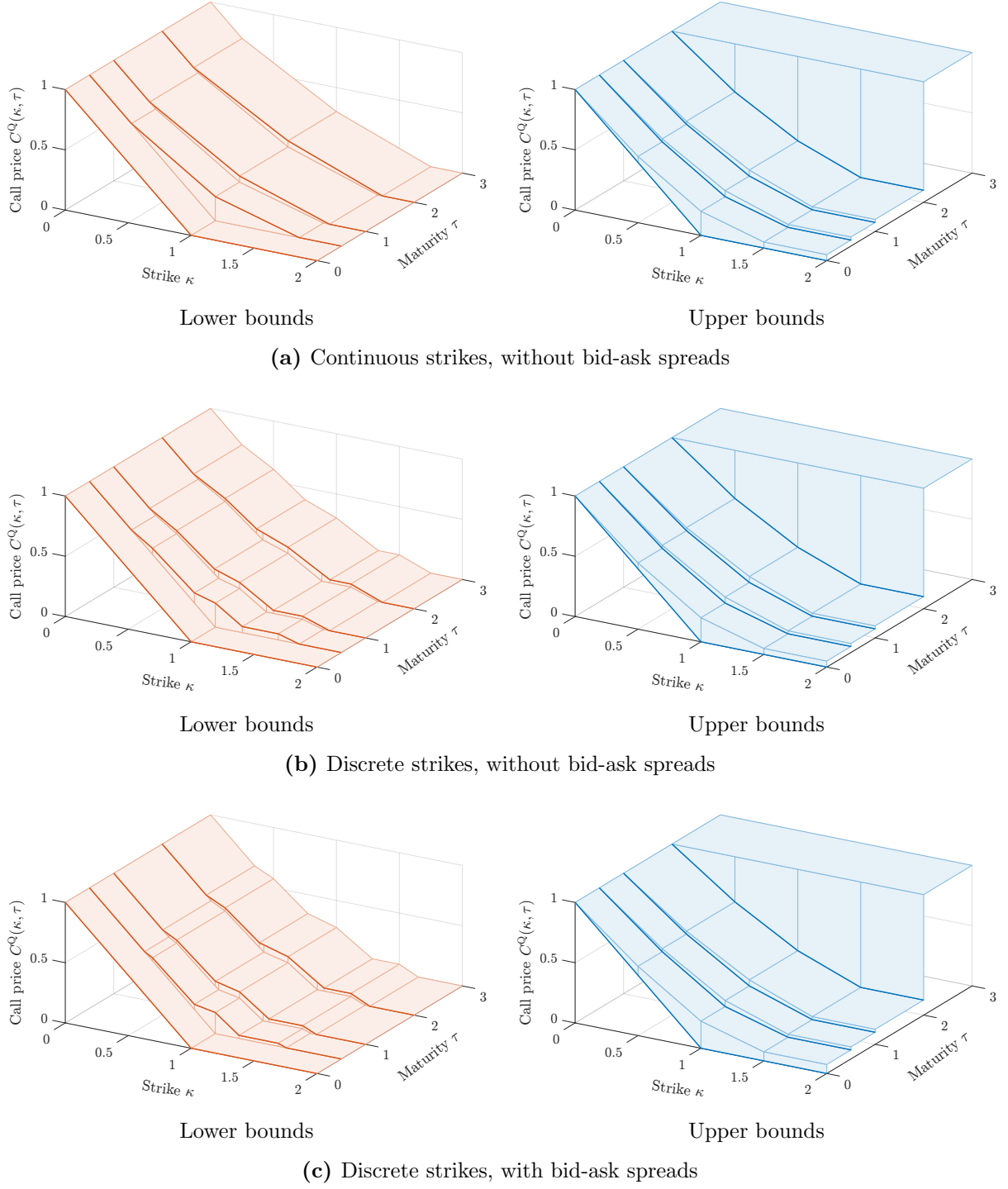


Figure OE.5: Illustrative option price bounds, multiple maturities

This figure illustrates the option price bounds in Proposition 3.1 within the settings of sections OE.4 to OE.6. Option prices are observed for multiple maturities $\mathbb{T} = \{0.5, 1, 2\}$. Mid prices $C(k, t)$ are specified as $C(0.5, 0.5) = 0.55$, $C(1, 0.5) = 0.2$, $C(1.5, 0.5) = 0.05$, $C(0.5, 1) = 0.575$, $C(1, 1) = 0.25$, $C(1.5, 1) = 0.075$, $C(0.5, 2) = 0.6$, $C(1, 2) = 0.3$, $C(1.5, 2) = 0.1$, otherwise using linear interpolation. Bid and ask option prices are determined as $B^C(k, t) = C(k, t) - s/2$ and $A^C(k, t) = C(k, t) + s/2$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ given a bid-ask spread s . In panel (a), we set $\mathbb{K} = [0.5, 1.5]$ and $s = 0$; in panel (b), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0$; in panel (c), we set $\mathbb{K} = \{0.5, 1, 1.5\}$ and $s = 0.05$. In each panel, the left and right plots show the lower bounds $\mathcal{L}^C(\kappa, \tau)$ and upper bounds $\mathcal{U}^C(\kappa, \tau)$, respectively.

OE.7 Continuous strikes, multiple maturities, without bid-ask spreads

As a variant of the setting in section OE.4, suppose again that option prices without bid-ask spreads are observed at multiple maturities $\mathbb{T} := \{t_1, \dots, t_m\}$ with strictly increasing $t_j > 0$. However, for each traded maturity $t \in \mathbb{T}$, now consider a maturity-dependent strike interval $\mathbb{K}_t := [k_*(t), k^*(t)] \subset (0, \bar{K})$, where $k_*(t) < 1 < k^*(t)$. Bid and ask prices are given by $B^C(k, t) = A^C(k, t) = C(k, t)$ for all $k \in \mathbb{K}_t$, $t \in \mathbb{T}$, such that $C(k, t)$ is an arbitrage-free call price surface and $P(k, t) = C(k, t) + k - 1$ is the associated put price surface.

It is straightforward to map this setting to one with a maturity-independent strike set by defining $\mathbb{K} := [\min_{t \in \mathbb{T}} k_*(t), \max_{t \in \mathbb{T}} k^*(t)]$ as well as $B^C(k, t) = (1 - k)^+$ and $A^C(k, t) = 1$ for all $k \notin \mathbb{K}_t$, $t \in \mathbb{T}$.

Option price bounds

Similar to section OE.4, we would like to apply Corollary OC.3 to obtain tractable expressions for the option price bounds in Proposition 3.1. However, unlike in the setting of section OE.4, Corollary OC.3 does not immediately apply. In order to enable its use, we additionally impose the assumption that

$$C((k \vee k_*(t')) \wedge k^*(t'), t') - \frac{C(k_*(t'), t') - 1}{k_*(t')} (k_*(t') - k)^+ \leq A^C(k, t)$$

for all $k \in \mathbb{K}_t$ and $t, t' \in \mathbb{T}$ with $t' \leq t$.

Under this additional assumption, the conditions of Corollary OC.3 are satisfied and yield similar expressions for the option price bounds as in section OE.4. To state the expressions, we again define $I^* := \min\{i : t_i \geq \tau\}$ and $I_* := \max\{i : t_i \leq \tau\}$ given a particular $\tau \geq 0$, using $t_0 = 0$ and $t_{m+1} = \infty$ with $C(k, 0) = (1 - k)^+$ and $C(k, \infty) = 1$. The call price bounds in equations (3.6) and (3.7) for $\tau \geq 0$ and $\kappa \geq 0$ can then be written as

$$\begin{aligned} \mathcal{U}^C(\kappa, \tau) &= C((\kappa \vee k_*(t_{I^*})) \wedge k^*(t_{I^*}), t_{I^*}) - \frac{C(k_*(t_{I^*}), t_{I^*}) - 1}{k_*(t_{I^*})} (k_*(t_{I^*}) - \kappa)^+, \\ \mathcal{L}^C(\kappa, \tau) &= (1 - \kappa)^+ \vee \bigvee_{i \leq I_*} \left(C((\kappa \vee k_*(t_i)) \wedge k^*(t_i), t_i) - \partial_{k+} C(k_*(t_i), t_i) (k_*(t_i) - \kappa)^+ \right. \\ &\quad \left. + \partial_{k-} C(k^*(t_i), t_i) (\kappa - k^*(t_i))^+ \right). \end{aligned}$$

Analogously, the put price bounds in equations (3.8) and (3.9) for $\tau \geq 0$ and $\kappa \geq 0$ can be expressed as

$$\begin{aligned} \mathcal{U}^P(\kappa, \tau) &= P((\kappa \vee k_*(t_{I^*})) \wedge k^*(t_{I^*}), t_{I^*}) - \frac{P(k_*(t_{I^*}), t_{I^*})}{k_*} (k_*(t_{I^*}) - \kappa)^+ + (\kappa - k^*(t_{I^*}))^+, \\ \mathcal{L}^P(\kappa, \tau) &= (\kappa - 1)^+ \vee \bigvee_{i \leq I_*} \left(P((\kappa \vee k_*(t_i)) \wedge k^*(t_i), t_i) - \partial_{k+} P(k_*(t_i), t_i) (k_*(t_i) - \kappa)^+ \right. \\ &\quad \left. + \partial_{k-} P(k^*(t_i), t_i) (\kappa - k^*(t_i))^+ \right). \end{aligned}$$

Figure OE.5(a) illustrates the surfaces of option price bounds $\mathcal{L}^\chi(\kappa, \tau)$ and $\mathcal{U}^\chi(\kappa, \tau)$ without the additional assumption, where one needs to resort to the general form of the bounds in Proposition 3.1.

OE.8 Discrete strikes, multiple maturities, without bid-ask spreads

As a variant of the setting in section OE.5, consider multiple maturities $\mathbb{T} := \{t_1, \dots, t_m\}$ with strictly increasing $t_j > 0$. At each maturity $t \in \mathbb{T}$, we have a maturity-dependent strike set $\mathbb{K}_t := \{k_1(t), \dots, k_{n(t)}(t)\} \subset (0, \bar{K})$ with $k_*(t) := k_1(t) < 1 < k_{n(t)}(t) =: k^*(t)$ and strictly increasing $k_i(t)$. Bid

and ask prices are given by $B^C(k, t) = A^C(k, t) = C(k, t)$ for all $k \in \mathbb{K}_t$, $t \in \mathbb{T}$, where $C(k, t)$ is some arbitrage-free call price surface and $P(k, t) = C(k, t) + k - 1$ is the associated put price surface.

We may straightforwardly map this setting to one with a maturity-independent strike set by defining $\mathbb{K} := \bigcup_{t \in \mathbb{T}} \mathbb{K}_t$ as well as $B^C(k, t) = (1 - k)^+$ and $A^C(k, t) = 1$ for all $k \notin \mathbb{K}_t$, $t \in \mathbb{T}$.

Option price bounds

Also in the present setting, the conditions of Corollary OC.3 are not automatically satisfied. We thus again need an additional assumption in order to obtain tractable expressions for the option price bounds in Proposition 3.1 via Corollary OC.3. In this regard, we suppose that

$$\frac{k_{j+1}(t') - k}{k_{j+1}(t') - k_j(t')} C(k_j(t'), t') + \frac{k - k_j(t')}{k_{j+1}(t') - k_j(t')} C(k_{j+1}(t'), t') \leq A^C(k, t)$$

for all $k \in \mathbb{K}_t$ with $k_j(t') \leq k < k_{j+1}(t')$ and $t, t' \in \mathbb{T}$ with $t' \leq t$, employing the usual augmentation of $\mathbb{K}_t$ by $k_0(t) = 0$ and $k_{n+1}(t) = \infty$ with $C(0, t) = 1$ and $C(\infty, t) = 0$ at any $t \in \mathbb{T}$.

Under this additional assumption, we may apply Corollary OC.3 to arrive at similar expressions as in section OE.5. For this, we again define $I^* := \min\{i : t_i \geq \tau\}$ and $I_* := \max\{i : t_i \leq \tau\}$ for a given $\tau \geq 0$, using $t_0 = 0$ and $t_{m+1} = \infty$ with $C(k, 0) = (1 - k)^+$ and $C(k, \infty) = 1$. The call price bounds in equations (3.6) and (3.7) for $\tau \geq 0$ and $k_j(t_i) < \kappa \leq k_{j+1}(t_i)$ can then be given as

$$\begin{aligned} \mathcal{U}^C(\kappa, \tau) &= \frac{k_{j+1}(t_{I^*}) - \kappa}{k_{j+1}(t_{I^*}) - k_j(t_{I^*})} C(k_j(t_{I^*}), t_{I^*}) + \frac{\kappa - k_j(t_{I^*})}{k_{j+1}(t_{I^*}) - k_j(t_{I^*})} C(k_{j+1}(t_{I^*}), t_{I^*}), \\ \mathcal{L}^C(\kappa, \tau) &= \bigvee_{i \leq I_*} \left(\frac{k_j(t_i) - \kappa}{k_j(t_i) - k_{j-1}(t_i)} C(k_{j-1}(t_i), t_i) + \frac{\kappa - k_{j-1}(t_i)}{k_j(t_i) - k_{j-1}(t_i)} C(k_j(t_i), t_i) \right) \\ &\quad \bigvee \left(\frac{k_{j+2}(t_i) - \kappa}{k_{j+2}(t_i) - k_{j+1}(t_i)} C(k_{j+1}(t_i), t_i) + \frac{\kappa - k_{j+1}(t_i)}{k_{j+2}(t_i) - k_{j+1}(t_i)} C(k_{j+2}(t_i), t_i) \right). \end{aligned}$$

The put price bounds in equations (3.8) and (3.9) for $\tau \geq 0$ and $k_j(t_i) \leq \kappa < k_{j+1}(t_i)$ obtain as

$$\begin{aligned} \mathcal{U}^P(\kappa, \tau) &= \frac{k_{j+1}(t_{I^*}) - \kappa}{k_{j+1}(t_{I^*}) - k_j(t_{I^*})} P(k_j(t_{I^*}), t_{I^*}) + \frac{\kappa - k_j(t_{I^*})}{k_{j+1}(t_{I^*}) - k_j(t_{I^*})} P(k_{j+1}(t_{I^*}), t_{I^*}), \\ \mathcal{L}^P(\kappa, \tau) &= \bigvee_{i \leq I_*} \left(\frac{k_j(t_i) - \kappa}{k_j(t_i) - k_{j-1}(t_i)} P(k_{j-1}(t_i), t_i) + \frac{\kappa - k_{j-1}(t_i)}{k_j(t_i) - k_{j-1}(t_i)} P(k_j(t_i), t_i) \right) \\ &\quad \bigvee \left(\frac{k_{j+2}(t_i) - \kappa}{k_{j+2}(t_i) - k_{j+1}(t_i)} P(k_{j+1}(t_i), t_i) + \frac{\kappa - k_{j+1}(t_i)}{k_{j+2}(t_i) - k_{j+1}(t_i)} P(k_{j+2}(t_i), t_i) \right). \end{aligned}$$

Figure OE.5(b) illustrates the surfaces of option price bounds $\mathcal{L}^X(\kappa, \tau)$ and $\mathcal{U}^X(\kappa, \tau)$ without the additional assumption, where one needs to resort to the general form of the bounds in Proposition 3.1.

OE.9 Discrete strikes, multiple maturities, with bid-ask spreads

As a variant of the setting in section OE.6, consider again multiple maturities $\mathbb{T} := \{t_1, \dots, t_m\}$ with strictly increasing $t_j > 0$. Each traded maturity $t \in \mathbb{T}$ is now associated with a maturity-dependent strike set $\mathbb{K}_t := \{k_1(t), \dots, k_{n(t)}(t)\} \subset (0, \bar{K})$ with $k_*(t) := k_1(t) < 1 < k_{n(t)}(t) =: k^*(t)$ and strictly increasing $k_i(t)$. Bid and ask prices are given by $(1 - k)^* \leq B^C(k, t) < C(k, t) < A^C(k, t) \leq 1$ for all $k \in \mathbb{K}_t$, $t \in \mathbb{T}$, where $C(k, t)$ is some arbitrage-free call price surface.

This setting can likewise be mapped to one with a maturity-independent strike set by defining $\mathbb{K} := \bigcup_{t \in \mathbb{T}} \mathbb{K}_t$ as well as $B^C(k, t) = (1 - k)^+$ and $A^C(k, t) = 1$ for all $k \notin \mathbb{K}_t$, $t \in \mathbb{T}$.

Option price bounds

To derive tractable expressions for the option price bounds in Proposition 3.1, we once more invoke Corollary OC.3. For this, we impose the additional assumption that

$$\frac{k'' - k}{k'' - k'} A^C(k', t') + \frac{k - k'}{k'' - k'} A^C(k', t') \leq A^C(k, t)$$

for all $k \in \mathbb{K}_t$ and $k', k'' \in \mathbb{K}_{t'} \cup \{0, \infty\}$ with $k' \leq k < k''$ as well as $t, t' \in \mathbb{T}$ with $t' \leq t$. As usual, we here use $C(0, t) = 1$ and $C(\infty, t) = 0$ at any $t \in \mathbb{T}$.

Under this additional assumption, Corollary OC.3 allows to express the option price bounds in Proposition 3.1 essentially in the form as in section OE.6.

Figure OE.5(c) illustrates the surfaces of option price bounds $\mathcal{L}^\chi(\kappa, \tau)$ and $\mathcal{U}^\chi(\kappa, \tau)$ without the additional assumption, where one needs to resort to the general form of the bounds in Proposition 3.1.

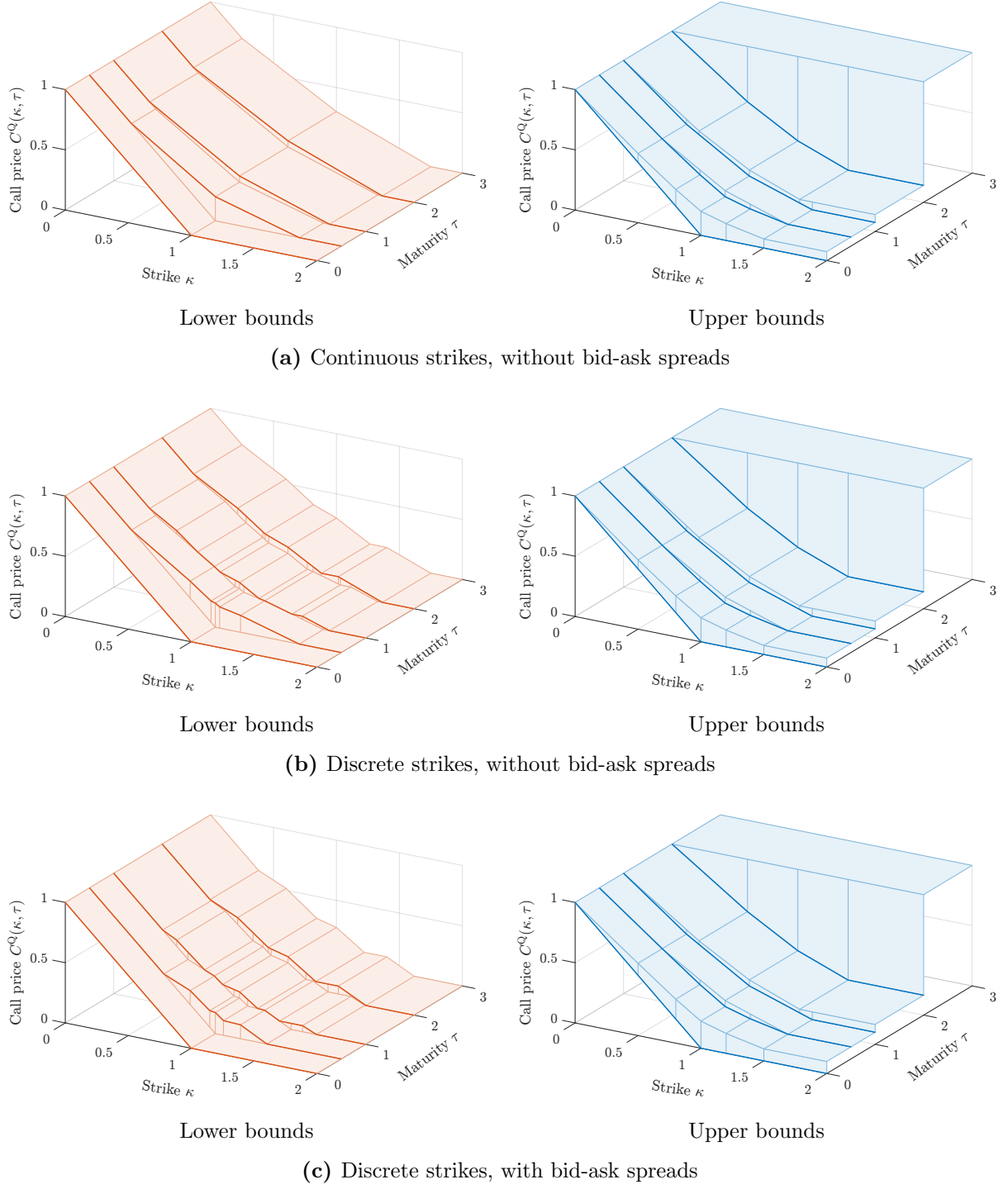


Figure OE.6: Illustrative option price bounds, multiple maturities

This figure illustrates the option price bounds in Proposition 3.1 within the settings of sections OE.7 to OE.9. Option prices are observed for multiple maturities $\mathbb{T} = \{0.5, 1, 2\}$. Mid prices $C(k, t)$ are specified as $C(0.8, 0.5) = 0.34$, $C(1, 0.5) = 0.2$, $C(1.2, 0.5) = 0.14$, $C(0.5, 1) = 0.575$, $C(1, 1) = 0.25$, $C(1.5, 1) = 0.075$, $C(0.6, 2) = 0.54$, $C(1, 2) = 0.3$, $C(1.4, 2) = 0.14$, otherwise using linear interpolation. Bid and ask option prices are determined as $B^C(k, t) = C(k, t) - s/2$ and $A^C(k, t) = C(k, t) + s/2$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ given a bid-ask spread s . In panel (a), we set $\mathbb{K}_{0.5} = [0.8, 1.2]$, $\mathbb{K}_1 = [0.5, 1.5]$, $\mathbb{K}_2 = [0.6, 1.4]$ and $s = 0$; in panel (b), we set $\mathbb{K}_{0.5} = \{0.8, 1, 1.2\}$, $\mathbb{K}_1 = \{0.5, 1, 1.5\}$, $\mathbb{K}_2 = \{0.6, 1, 1.4\}$ and $s = 0$; in panel (c), we set $\mathbb{K}_{0.5} = \{0.8, 1, 1.2\}$, $\mathbb{K}_1 = \{0.5, 1, 1.5\}$, $\mathbb{K}_2 = \{0.6, 1, 1.4\}$ and $s = 0.05$. In each panel, the left and right plots show the lower bounds $\mathcal{L}^C(\kappa, \tau)$ and upper bounds $\mathcal{U}^C(\kappa, \tau)$, respectively.

OF Numerical implementation and experiments

OF.1 Algorithms

In this section, we provide algorithms that implement the option price bounds in Proposition 3.1 and tail probability bounds in Proposition 3.3. To show selected details of these algorithms, we use Matlab-style pseudo-code, in which we prioritize comprehensibility over syntactic correctness.

OF.1.1 Option price bounds

With regards to Proposition 3.1, it suffices to provide an algorithm for the call price bounds in equations (3.6) and (3.7). The respective put price bounds in equations (3.8) and (3.9) then follow using the put-call parity relations (3.5). Due to equation (3.12), it further suffices to obtain the option price bounds at the traded maturities in $\mathbb{T}$, which then entirely determine the bounds at any maturity $\tau \geq 0$.

Step 0: To accommodate option data as observed in empirical markets, suppose that bid and ask prices $(1 - k)^+ \leq B^C(k, t) \leq A^C(k, t) \leq 1$ are given for $k \in \mathbb{K}_t \subset (0, \infty)$, $t \in \mathbb{T} \subset (0, \infty)$, where all $\mathbb{K}_t$ and $\mathbb{T}$ are finite sets. Concretely, we have $\mathbb{K}_t := \{k_1(t), \dots, k_{n(t)}(t)\}$ with $k_i(t)$ in strictly increasing order. Without loss of generality, since only the ordering of maturities matters, we further take $\mathbb{T} = \{1, \dots, T\}$. In light of Corollaries OC.3 and OC.4, we build our algorithm on the pooled bid-ask prices

$$\tilde{B}^C(k, t) := \max\{B^C(k, t') : t' \leq t\}, \quad \tilde{A}^C(k, t) := \min\{A^C(k, t') : t' \geq t\}$$

for all $k \in \tilde{\mathbb{K}}_t^B$, $t \in \mathbb{T}$ and $k \in \tilde{\mathbb{K}}_t^A$, $t \in \mathbb{T}$, respectively, where we may set $\tilde{\mathbb{K}}_t^B := \mathbb{K}_t \cup \{0, \infty\} \cup \bigcup_{t' < t} \mathbb{K}_{t'}$ and $\tilde{\mathbb{K}}_t^A := \mathbb{K}_t \cup \{0, \infty\} \cup \bigcup_{t' > t} \mathbb{K}_{t'}$ with $B^C(0, t) = A^C(0, t) = 1$ and $B^C(\infty, t) = A^C(\infty, t) = 0$. In fact, it suffices to choose $\tilde{\mathbb{K}}_t^B$ and $\tilde{\mathbb{K}}_t^A$ smaller by including only those strikes in $\mathbb{K}_{t'}$ at $t' \neq t$ that may affect the respective bounds.

For bids, we thus initialize $B(t).P$ using $B^C(k, t)$ and $B(t).K$ using $\mathbb{K}_t \cup \{0, \infty\}$. Subsequently, we perform the following updates to essentially arrive at more efficient versions of $\tilde{B}^C(k, t)$ and $\tilde{\mathbb{K}}_t^B$:

```

for t = 1 : T
    B(t).P = cummax(B(t).P, direction='backward');
    if t > 2
        I = B(t-1).P > interp1(B(t).K, B(t).P, B(t-1).K, method='next');
        B(t).P = [B(t).P; B(t-1).P(I)];
        B(t).K = [B(t).K; B(t-1).K(I)];
    end
end

```

For asks, we proceed similarly by initializing $A(t).P$ using $A^C(k, t)$ and $A(t).K$ using $\mathbb{K}_t \cup \{0, \infty\}$. We then perform the following updates to obtain more efficient versions of $\tilde{A}^C(k, t)$ and $\tilde{\mathbb{K}}_t^A$:

```

for t = T : -1 : 1
    A(t).P = cummin(A(t).P, direction='forward');
    if t < T
        I = A(t+1).P < interp1(A(t).K, A(t).P, A(t+1).K, method='linear');
        A(t).P = [A(t).P; A(t+1).P(I)];
        A(t).K = [A(t).K; A(t+1).K(I)];
    end
end

```

Finally, we restore the strictly increasing order in all $B(t).K$ and $A(t).K$.

Upper bound

Provided feasibility, the upper bounds in equation (3.6) at $\tau = t$ and the strikes in $\tilde{\mathbb{K}}_t^A$ are determined by sequentially updating $U^C(k, t)$. We initialize $U^C(k, t) = \tilde{A}^C(k, t)$ for all $k \in \tilde{\mathbb{K}}_t^A$, $t \in \mathbb{T}$ by setting $U(t).P = A(t).P$ and $U(t).K = A(t).K$.

Step U1: In the first step, we ensure that $U^C(k, t)$ is decreasing in k in the sense that

$$U^C(k_i(t), t) = \min\{U^C(k_j(t), t) : j \leq i\}.$$

This can be implemented via the following cumulative minimum:

```

for t = 1 : T
    U(t).P = cummin(U(t).P, direction='forward');
end

```

Step U2: In the second step, $U^C(k, t)$ is made convex in k such that it satisfies

$$U^C(k_i(t), t) = \min\left\{\frac{k_{j'}(t) - k_i(t)}{k_{j'}(t) - k_j(t)} U^C(k_j(t), t) + \frac{k_i(t) - k_j(t)}{k_{j'}(t) - k_j(t)} U^C(k_{j'}(t), t) : j \leq i < j'\right\}.$$

It should be noted that due to the employed construction, it here suffices to consider only the maturity t . To establish this property iteratively for all i , only the two choices $j = i - 1$ and $j = i$ need to be accounted for. Therefore, we update $U^C(k, t)$ as follows:

```

for t = 1 : T
    for i = 2 : numel(U(t).K)-1
        D = min((U(t).P(i+1:end)-U(t).P(i-1)) ./ (U(t).K(i+1:end)-U(t).K(i-1)));
        U(t).P(i) = min(U(t).P(i), U(t).P(i-1)+D*(U(t).K(i)-U(t).K(i-1)));
    end
end

```

After these updates, the value stored in $U(t).P(i)$ equals the upper bound $\mathcal{U}^C(k_i(t), t)$ in equation (3.6) subject to the bid-ask constraints $B^C(k, t')$ and $A^C(k, t')$ for $k \in \mathbb{K}_{t'}$, $t' \in \mathbb{T}$.

Step U3: In the final step, it only remains to obtain the upper bound in piecewise-linear functional form. For this, we simply define $\text{Ufcn}\{t\}$ as the linear interpolant of $U(t).P$ at the strikes $U(t).K$. Thereby, we now have that the evaluation of the linear interpolant $\text{Ufcn}\{t\}(\kappa)$ equals $\mathcal{U}^C(\kappa, t)$ for any $\kappa \geq 0$. To incorporate also the cases $\tau = 0$ and $\tau > T$, we further set $\text{Ufcn}\{0\}(\kappa)$ equal to $(1 - \kappa)^+$ and $\text{Ufcn}\{T+1\}(\kappa)$ equal to 1.

Lower bound

Provided feasibility, the lower bounds in equation (3.7) at $\tau = t$ and the strikes in $\widetilde{\mathbb{K}}_t^B$ are determined by sequentially updating $L^C(k, t)$. We initialize $L^C(k, t) = \widetilde{B}^C(k, t)$ for all $k \in \widetilde{\mathbb{K}}_t^B$, $t \in \mathbb{T}$ by setting $L(\mathbf{t}).P = B(\mathbf{t}).P$ and $L(\mathbf{t}).K = B(\mathbf{t}).K$. Alongside the prices $L(\mathbf{t}).P$, we also store and update the left-hand and right-hand directional derivatives $L(\mathbf{t}).DL$ and $L(\mathbf{t}).DR$, which we however do not make explicit in what follows.

Step L1: In the first step, we likewise ensure that $L^C(k, t)$ is decreasing in k , so that

$$L^C(k_i(t), t) = \max\{L^C(k_j(t), t) : j \geq i\}.$$

This can be implemented as the following cumulative maximum in reverse order:

```

for t = 1 : T
    L(t).P = cummax(L(t).P, direction='backward');
end

```

Step L2: In the second step, we now incorporate the two main extrapolation expressions in equation (3.7). While the order of the two does not matter, we start by ensuring that

$$L^C(k_i(t), t) = \max\left\{\frac{k_{j'}(t) - k_i(t)}{k_{j'}(t) - k_j(t)} \mathcal{U}^C(k_j(t), t) + \frac{k_i(t) - k_j(t)}{k_{j'}(t) - k_j(t)} L^C(k_{j'}(t), t) : j < j' \leq i\right\},$$

using the upper bounds $\mathcal{U}^C(k_j(t), t)$, whose construction allows to consider just the maturity t . Iteratively over i , it suffices to take into account the two choices $j' = i - 1$ and $j' = i$. Hence, we may use the following implementation:

```

for t = 1 : T
    for i = 2 : numel(L(t).K)
        I = U(t).K < L(t).K(i-1);
        D = max((L(t).P(i-1)-U(t).P(I))./(L(t).K(i-1)-U(t).K(I)));
        L(t).P(i) = max(L(t).P(i), L(t).P(i-1)+D*(L(t).K(i)-L(t).K(i-1)));
    end
end

```

Next, we continue by enforcing that

$$L^C(k_i(t), t) = \max\left\{\frac{k_{j'}(t) - k_i(t)}{k_{j'}(t) - k_j(t)} L^C(k_j(t), t) + \frac{k_i(t) - k_j(t)}{k_{j'}(t) - k_j(t)} \mathcal{U}^C(k_{j'}(t), t) : i \leq j < j'\right\},$$

again using the upper bounds $\mathcal{U}^C(k_{j'}(t), t)$ and the single maturity t . When iterating over i in reverse order, it now suffices to account for the two choices $j = i$ and $j = i + 1$. Therefore, we utilize the following implementation:

```

for t = 1 : T
    for i = numel(L(t).K)-1 : -1 : 1
        I = U(t).K > L(t).K(i+1);
        D = min((U(t).P(I)-L(t).P(i+1))./(U(t).K(I)-L(t).K(i+1)));
        L(t).P(i) = max(L(t).P(i), L(t).P(i+1)+D*(L(t).K(i)-L(t).K(i+1)));
    end
end

```

After these updates, $L(t).P(i)$ equals the lower bound $\tilde{\mathcal{L}}^C(k_i(t), t)$ as in equation (3.7) when considering the bid-ask constraints $\tilde{B}^C(k, t)$ and $\tilde{A}^C(k, t)$ for $k \in \tilde{\mathbb{K}}_t^B \cup \tilde{\mathbb{K}}_t^A$ at the single maturity t .

Step L3: In the final step, we first use the prices $L(t).P$ together with the directional derivatives $L(t).DL$ and $L(t).DR$ to obtain the piecewise-linear function $Lfcn\{t\}$. For the pointwise maximum of these functions, we have the relation $\mathcal{L}^C(\kappa, t) = \bigvee_{t' \leq t} \tilde{\mathcal{L}}^C(\kappa, t')$. Therefore, we determine iteratively the following functional maxima:

```

for t = T-1 : -1 : 1
    Lfcn{t} = max(Lfcn{t}, Lfcn{t+1});
end

```

Notably, each pointwise maximum of piecewise-linear functions can be straightforwardly obtained as another piecewise-linear function. Finally, we thereby have that the evaluation $Lfcn\{t\}(\kappa)$ equals $\mathcal{L}^C(\kappa, t)$ in equation (3.7) for any $\kappa \geq 0$. The case $0 \leq \tau < 1$ is accommodated by setting $Lfcn\{0\}(\kappa)$ equal to $(1 - \kappa)^+$.

OF.1.2 Tail probability bounds

Given the option price bounds in functional form according to section OF.1.1, we can determine the associated tail probability bounds in Proposition 3.3. In what follows, we provide algorithms to compute these bounds for some given strike $\kappa \geq 0$ and maturity $\tau \geq 0$.

Throughout, exploiting equation (3.12) and the outputs produced in section OF.1.1, we use the following definitions:

```

t_U = min(ceil(τ), T+1);    t_L = min(floor(τ), T);
I_L = U(t_U).K < κ;        I_R = U(t_U).K > κ;
k_L = U(t_U).K(I_L);       k_R = U(t_U).K(I_R);

```

Here, recall that $\mathbb{T} = \{1, \dots, T\}$ and $U(t).K$ includes $\{0, \infty\}$ at any $t \in \mathbb{T}$. In addition, we set $U(0).K$ equal to $\{0, 1, \infty\}$ to also cover the case where $0 \leq \tau < 1$ as well as $U(T+1).K$ equal to $\{0, \infty\}$ for the case $\tau > T$.

Left-tail probability, upper bound

For the upper bound $\mathcal{U}^{LTP}(\kappa, \tau)$ in equation (3.22), the put-call parity relations (3.5) immediately suggest the following implementation:

```

U_LTP = min(1+(Ufcn{t_U}(k_R)-Lfcn{t_L}(κ))./(k_R-κ));

```

Left-tail probability, lower bound

Regarding the lower bound $\mathcal{L}^{LTP}(\kappa, \tau)$ in equation (3.23), we first note that with finitely many strikes in $U(t).K$, we have the equivalent representation

$$\mathcal{L}^{LTP}(\kappa, \tau) = \left(\partial_{k+} \mathcal{L}^P(\kappa, \tau) + \lim_{\eta \downarrow 0} \frac{\mathcal{L}^P(\kappa, \tau) - \mathcal{U}^P(\kappa, \tau)}{\eta} \right) \vee \max \left\{ \frac{\mathcal{L}^P(\kappa, \tau) - \mathcal{U}^P(k', \tau)}{\kappa - k'} : k' < \kappa \right\}.$$

Notably, the term in parentheses equals $\partial_{k+} \mathcal{L}^P(\kappa, \tau)$ if $\mathcal{L}^P(\kappa, \tau) = \mathcal{U}^P(\kappa, \tau)$ and otherwise $-\infty$. Moreover, in the former case, this term yields the lower bound.

Due to the piecewise-linear form of $Lfcn\{t\}$, it is straightforward to determine the associated right-hand derivative $Lfcn_DR\{t\}$ in piecewise-constant functional form. By the above reasoning, we therefore arrive at the following implementation of the lower bound $\mathcal{L}^{LTP}(\kappa, \tau)$:

```

if Ufcn{t_U}(κ) == Lfcn{t_L}(κ)
    L_LTP = 1+Lfcn_DR{t_L}(κ);
else
    L_LTP = max(1+(Lfcn{t_L}(κ)-Ufcn{t_U}(κ_L))./(κ-κ_L));
end

```

Right-tail probability, upper bound

The upper bound $\mathcal{U}^{RTP}(\kappa, \tau)$ in equation (3.24) can be directly computed as follows:

```

U_RTP = min(-(Lfcn{t_L}(κ)-Ufcn{t_U}(κ_L))./(κ-κ_L));

```

Right-tail probability, lower bound

For the lower bound $\mathcal{L}^{RTP}(\kappa, \tau)$, note that for finitely many strikes in $\mathbf{U}(\mathbf{t}).\mathbf{K}$, equation (3.25) can be expressed equivalently as

$$\mathcal{L}^{RTP}(\kappa, \tau) = 0 \vee \left(-\partial_{\kappa-} \mathcal{L}^C(\kappa, \tau) - \lim_{\eta \downarrow 0} \frac{\mathcal{U}^C(\kappa, \tau) - \mathcal{L}^C(\kappa, \tau)}{\eta} \right) \vee \max \left\{ -\frac{\mathcal{U}^C(k', \tau) - \mathcal{L}^C(\kappa, \tau)}{k' - \kappa} : \kappa < k' \right\}.$$

The term in parentheses equals $-\partial_{\kappa-} \mathcal{L}^C(\kappa, \tau)$ if $\mathcal{L}^C(\kappa, \tau) = \mathcal{U}^C(\kappa, \tau)$ and otherwise $-\infty$. In the former case, this term yields the lower bound.

From the piecewise-linear form of $\mathbf{Lfcn}\{\mathbf{t}\}$, we can straightforwardly obtain the associated left-hand derivative $\mathbf{Lfcn_DL}\{\mathbf{t}\}$ in piecewise-constant functional form. This yields the following implementation of the lower bound $\mathcal{L}^{RTP}(\kappa, \tau)$:

```

if Ufcn{t_U}(κ) == Lfcn{t_L}(κ)
    L_RTP = -Lfcn_DL{t_L}(κ);
else
    L_RTP = max(-(Ufcn{t_U}(κ_R)-Lfcn{t_L}(κ))./(κ_R-κ));
end

```

OF.2 Numerical programs

As a benchmark for the exact algorithms in section OF.1, we now formulate a general linear program that can be used to (approximately) determine the option price bounds in Proposition 3.1 and tail probability bounds in Proposition 3.3. We again assume that bid and ask prices $(1-k)^+ \leq B^C(k, t) \leq A^C(k, t) \leq 1$ are observed for $k \in \mathbb{K}_t \subset (0, \infty)$, $t \in \mathbb{T} \subset (0, \infty)$, where all $\mathbb{K}_t$ and $\mathbb{T}$ are finite sets. Without loss of generality, maturities are given as $\mathbb{T} = \{1, \dots, T\}$ such that also $\tau \in \mathbb{T}$. Moreover, the joint strike set is denoted by $\mathbb{K} := \bigcup_{t \in \mathbb{T}} \mathbb{K}_t := \{k_1, \dots, k_n\}$ with strictly increasing k_i .

To simplify the discussion, consider for a generic payoff $f(R_\tau)$ the problems

$$\inf_{\mathbb{Q} \in \mathcal{O}(\mathcal{T})} / \sup_{\mathbb{Q} \in \mathcal{O}(\mathcal{T})} \mathbb{E}^{\mathbb{Q}_\tau}[f(R_\tau)]. \quad (\text{OF.1})$$

Among others, the option price problems (3.4) and tail probability problems (3.21) are special cases of problems (OF.1) for appropriate choices of f .

Take a state discretization $\mathbb{X} := \{x_0 := 0, x_1, \dots, x_m, x_{m+1} := \infty\}$ with strictly increasing x_i in order to formulate finite-dimensional linear programs that approximate the infinite-dimensional linear programs (OF.1). Extending the single-period setup in Bondarenko et al. (2024), we thus define the

following linear programs on $\mathbb{X}$:

$$\begin{aligned}
\min / \max_{(q_{i,t})} \quad & \sum_{i=0}^m q_{i,\tau} f(x_i) + q_{m+1,\tau} \left(\lim_{u \rightarrow \infty} \frac{f(u)}{u} \right) \\
\text{s.t.} \quad & q_{j,t} \geq 0, \quad \sum_{i=0}^m q_{i,t} = 1, \quad \sum_{i=0}^m q_{i,t} x_i + q_{m+1,t} = 1, \\
& \sum_{i=0}^m q_{i,t} (x_i - x_j)^+ + q_{m+1,t} \leq \sum_{i=0}^m q_{i,t+1} (x_i - x_j)^+ + q_{m+1,t+1}, \\
& B^C(k, t) \leq \sum_{i=0}^m q_{i,t} (x_i - k)^+ + q_{m+1,t} \leq A^C(k, t),
\end{aligned} \tag{OF.2}$$

in which the no-arbitrage constraints hold for all $j \in \{0, \dots, m+1\}$ and $t \in \mathbb{T}$, while the bid-ask constraints are enforced for all $k \in \mathbb{K}_t$, $t \in \mathbb{T}$.

To guarantee the feasibility of problems (OF.2) whenever $\mathcal{O}(\mathcal{T}) \neq 0$, it suffices to have the inclusion $\mathbb{K} \subset \mathbb{X}$. The following lemma theoretically justifies the problems (OF.2) by explicitly linking each admissible $(q_{i,t})$ to a limit of feasible probability measures in $\mathcal{O}(\mathbb{T})$, which can be straightforwardly extended to $\mathcal{O}(\mathcal{T})$. Hence, minima and maxima obtained from problems (OF.2) will be contained within the range of the lower and upper bounds from problems (OF.1). A suitable choice of $\mathbb{X}$ tailored to the payoff function f moreover typically ensures a close approximation of these bounds.

Lemma OF.1. *Each feasible solution to problems (OF.2) on $\mathbb{X} := \{x_0 := 0, x_1, \dots, x_m, x_{m+1} := \infty\} \supset \mathbb{K}$ corresponds to a limit of discrete probability measures in $\mathcal{O}(\mathbb{T})$, supported on some $\mathbb{X}_u := \{x_0, x_1, \dots, x_m, x_{m+1} := u\} \subset \Omega$ for large enough u .*

Proof. Take $u > x_m$ large enough such that $B^C(k, t) \leq \mathcal{E}(k, t; \tilde{\mathbb{A}}_0(u)) \leq A^C(k, t)$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$ with $\tilde{\mathbb{A}}_0(u)$ as in Lemma OC.2. Setting $F(k, t) := \mathcal{E}(k \wedge x_m, t; \tilde{\mathbb{A}}_0(u))$, define $q_{i,t}^* := \partial_{k+} F(x_i, t) - \partial_{k-} F(x_i, t)$ for $i = 0, \dots, m$ and $q_{m+1,t}^* := F(x_m, t)$. By construction, $(q_{i,t}^*)$ is a feasible solution to problems (OF.2) on $\mathbb{X}$ with the property that $q_{m,t}^* > 0$ whenever $q_{m+1,t}^* > 0$.

Consider any feasible solution $(q_{i,t})$ to problems (OF.2) on $\mathbb{X}$. Exploiting that the constraints in problems (OF.2) define a convex feasible set, define another feasible solution $(\tilde{q}_{i,t}^\lambda)$ for any $0 < \lambda < 1$ through the convex combination $\tilde{q}_{i,t}^\lambda := \lambda q_{i,t}^* + (1 - \lambda) q_{i,t}$. For large enough u , $(\tilde{q}_{i,t}^\lambda)$ is associated to a probability measure $\tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda} \in \mathcal{O}(\mathbb{T})$ with point mass at locations $\{x_0, x_1, \dots, x_m, u\}$, given by

$$\tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda}(R_t = x_i) := \tilde{q}_{i,t}^\lambda, \quad \tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda}(R_t = x_m) := \tilde{q}_{m,t}^\lambda - \frac{\tilde{q}_{m+1,t}^\lambda}{u - x_m}, \quad \tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda}(R_t = u) := \frac{\tilde{q}_{m+1,t}^\lambda}{u - x_m}.$$

Each such $\tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda}$ satisfies by construction that $C^{\tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda}}(k, t) = \sum_{i=0}^m \tilde{q}_{i,t}^\lambda (x_i - k)^+ + \tilde{q}_{m+1,t}^\lambda$ for all $(k, t) \in \mathbb{K} \times \mathbb{T}$. Moreover, as $\lambda \rightarrow 0$ and $u \rightarrow \infty$ while u remains large enough, it further holds for any payoff $f(R_t)$ and every $t \in \mathbb{T}$ that

$$\mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{T}}^{u,\lambda}}[f(R_t)] \rightarrow \sum_{i=0}^m q_{i,t} f(x_i) + q_{m+1,t} \left(\lim_{u \rightarrow \infty} \frac{f(u)}{u} \right),$$

as in problems (OF.2). □

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